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From physics to interrupt handlers: the real to float step

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Presentation at Deducteam

November 20, 2012
Different levels of description

In control engineering, work on different levels to design and build a control system:

- **Format/high-level aspects**: system conception, modeling, possibly proof.
- **Concrete/low-level aspects**: creation of an object implementing the system.

Quadricopter, DRONE Project, MINES ParisTech & ECP.
Formal aspect

System definition:

- **Inputs**: sensors [accelerometer, sonar...] + references [operator instructions].
  
- **Outputs**: actions to act on environment [rotors rotation speed].

- Modeling in the form of equations to express relations between inputs and outputs: transfer functions or differential equations.
Formal aspect

System definition:
- **Inputs**: sensors [accelerometer, sonar...] + references [operator instructions].
- **Outputs**: actions to act on environment [rotors rotation speed].
- Modeling in the form of equations to express relations between inputs and outputs: transfer functions or differential equations.

System requirements:
- **Stability** conditions [bounded rotation speed].
- **Pursuit** of reference input [try to reach the ordered position].
- **Perturbation** rejection [wind].
Concrete aspect

Creation of a real object implementing the system.

- **Electronic circuit** that physically computes the transfer function.
- With a **microcontroller**: a small system with processor, memory, I/O devices, that runs a **program** implementing the transfer function.

[ATMEGA128
Frequency: 16 MHz
RAM: 4 KB
Prog. mem.: 128 KB]
Semantic gap

Antagonism:
- Abstract, mathematical model.
- Microcontroller code: program written in C, then compiled. Long (thousands of LoC), low-level (elementary operations, hardware management, interruptions).
Semantic gap

Antagonism:
- Abstract, mathematical model.
- Microcontroller code: program written in C, then compiled.

Series of transformations to go from abstract model to microcontroller code.
Semantic gap

Antagonism:
- Abstract, mathematical model.
- Microcontroller code: program written in C, then compiled.

Series of transformations to go from abstract model to microcontroller code.

Problem of proof transposition: Considering a model with correction proofs [stability], how to transpose down these proofs at the code level?

Interest: formally check the code, not only the model.

Difficulties: semantic gap, non-equivalent transformations (⇒ proofs must be checked).
Control-theoretical aspects

Produce a pseudocode from the abstract model:

- Solve the model differential equation, get a transfer function. (Laplace transform/Z transform, initial conditions problem.)
- If continuous-time model, discretization. (Problems with sampling, execution times.)

while transposing the proof.

Usual problems in control engineering.

Once done, discrete-time system with a loop on the transfer function $\Rightarrow$ pseudocode [in MATLAB]. Proof: invariants on this code.
Compilation aspects

At the other end: compilation of C code to machine code.

Risks of error:

- Important changes in the code: elementary operations, management of registers and of memory stack, instruction jumps.
- Possible optimizations.

Solutions:

- “Existing C compilers are reliable enough.”
- Proof-preserving compilation [Barthe].
- Certified compilation [CompCert].
What’s between?

Opener question. Several challenges:

1. High-level mathematical operations $\leadsto$ series of elementary instructions [matrices, sinus].

2. Real values $\leadsto$ machine words with limited precision.

3. On a microcontroller, data/events acquisition raises interruptions (real-time answer, energy consumption) $\Rightarrow$ particular code structure.
Example system

Very simple, linear invariant system.

Pseudocode:

\[
\begin{align*}
Ac &= \begin{bmatrix} 0.4990 & -0.0500; 0.0100, 1.0000 \end{bmatrix}; \\
Bc &= [1;0]; \\
Cc &= [564.48, 0]; \\
Dc &= -1280;
\end{align*}
\]

\[
xc = \text{zeros}(2,1);
\]

\[
\begin{align*}
\text{receive}(y,2); \text{receive}(yd,2); \\
\text{while } 1 \\
\quad yc &= \max(\min(y - yd,1),-1); \\
\quad u &= Cc*xc + Dc*yc; \\
\quad xc &= Ac*xc + Bc*yc; \\
\quad \text{send}(u,1); \\
\quad \text{receive}(y,2); \\
\quad \text{receive}(yd,2); \\
\text{end}
\end{align*}
\]

\[
\begin{align*}
x_c &= \begin{bmatrix} x_{c1} \ x_{c2} \end{bmatrix} \in \mathbb{R}^2: \text{controller state} \\
y \in \mathbb{R}: \text{reference input}; \ y_d \in \mathbb{R}: \text{real position} \\
y_c \in [-1,1]: \text{bounded gap} \\
u \in \mathbb{R}: \text{action to be performed}
\end{align*}
\]
Lyapunov stability

Lyapunov stability: all reachable states $x_c$ start near an equilibrium point $x_e$ and stay in a neighborhood $V$ of $x_e$ forever.

$V$ found solving a Lyapunov equation. On linear systems, $V$ is generally an ellipsoid.

Here, show that $x_c = \begin{pmatrix} x_{c_1} \\ x_{c_2} \end{pmatrix}$ belongs to the ellipse:

$$\mathcal{E}_P = \{ x \in \mathbb{R}^2 \mid x^T \cdot P \cdot x \leq 1 \}, \quad P = 10^{-3} \begin{pmatrix} 0.6742 & 0.0428 \\ 0.0428 & 2.4651 \end{pmatrix}. $$

$$x_c \in \mathcal{E}_P \iff 0.6742x_{c_1}^2 + 0.0856x_{c_1}x_{c_2} + 2.4651x_{c_2}^2 \leq 1000.$$
Stability proof

\( xc = \text{zeros}(2,1); \)
\( xc \in \mathcal{E}_P \)
\( \text{receive}(y,2); \text{receive}(yd,2); \)
\( xc \in \mathcal{E}_P \)
\( \text{while} \ 1 \)
\( \quad xc \in \mathcal{E}_P \)
\( yc = \max(\min(y - yd,1),-1); \)
\( xc \in \mathcal{E}_P, \ y_c^2 \leq 1 \)
\( \begin{bmatrix} x_c \\ y_c \end{bmatrix} \in \mathcal{E}_{Q_\mu} \quad | \quad Q_\mu = \begin{pmatrix} \mu P & 0_{2 \times 1} \\ 0_{1 \times 2} & 1 - \mu \end{pmatrix}, \mu = 0.9991 \)
\( u = Cc*xc + Dc*yc; \)
\( \begin{bmatrix} x_c \\ y_c \end{bmatrix} \in \mathcal{E}_{Q_\mu} \)
\( xc = Ac*xc + Bc*yc; \)
\( xc \in \mathcal{E}_{\tilde{P}} \quad | \quad \tilde{P} = \left[ (A_c \ B_c) \cdot Q_\mu^{-1} \cdot (A_c \ B_c)^T \right]^{-1} \)
\( \text{send}(u,1); \)
\( xc \in \mathcal{E}_{\tilde{P}} \)
\( \text{receive}(y,2); \)
\( xc \in \mathcal{E}_{\tilde{P}} \)
\( \text{receive}(yd,2); \)
\( xc \in \mathcal{E}_{\tilde{P}} \)
\( xc \in \mathcal{E}_P \)
\( xc \in \mathcal{E}_P \)
\( \text{end} \)

Proof given as code invariants

Implication (weakening) if two consecutive invariants.

Trivial, or easy to check with matrix computations.

Last implication closes the loop. Its validity relies on parameters \( A_c, B_c, C_c, D_c, \mu \): numerical verification needed.
Digression: with C instructions

High level mathematical operations $\leadsto$ series of scalar elementary instructions.
Here, matrix operations are expanded: the instruction

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$$

$$x_c = A_c x_c + B_c y_c;$$

$$x_c \in \mathcal{E}_{\tilde{P}} \quad | \quad \tilde{P} = \left[ \begin{pmatrix} A_c & B_c \end{pmatrix} \cdot Q_\mu^{-1} \cdot \begin{pmatrix} A_c & B_c \end{pmatrix}^T \right]^{-1}$$

becomes:

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu}$$

$$xb[0] = x_c[0];$$  \hspace{1cm} xb: buffer

$$xb[1] = x_c[1];$$

$$x_c[0] = A_c[0][0] \times xb[0] + A_c[0][1] \times xb[1] + y_c;$$

$$x_c[1] = A_c[1][0] \times xb[0] + A_c[1][1] \times xb[1];$$

???
Digression: with C instructions

High level mathematical operations $\leadsto$ series of scalar elementary instructions.

Here, matrix operations are expanded: the instruction

$$
\begin{pmatrix}
x_c \\
y_c
\end{pmatrix} \in \mathcal{E}_{Q_\mu}
$$

$$
x_c = A_c x_c + B_c y_c;
$$

$$
x_c \in \mathcal{E}_{\tilde{P}} \quad | \quad \tilde{P} = \left[ \left( A_c \quad B_c \right) \cdot Q_\mu^{-1} \cdot \left( A_c \quad B_c \right)^T \right]^{-1}
$$

becomes:

$$
\begin{pmatrix}
x_c \\
y_c
\end{pmatrix} \in \mathcal{E}_{Q_\mu}
$$

$$
x_{b[0]} = x_c[0];
$$

$$
x_{b[1]} = x_c[1];
$$

$$
x_{c[0]} = A_c[0][0] \cdot x_{b[0]} + A_c[0][1] \cdot x_{b[1]} + y_c;
$$

$$
x_{c[1]} = A_c[1][0] \cdot x_{b[0]} + A_c[1][1] \cdot x_{b[1]};
$$

$$
x_c \in \mathcal{E}_{\tilde{P}} \quad | \quad \tilde{P} = \left[ \left( A_c \quad B_c \right) \cdot Q_\mu^{-1} \cdot \left( A_c \quad B_c \right)^T \right]^{-1}
$$

Same computation: output invariant can be found [Feron].
Numerical precision problems

To produce C code: real numbers \leadsto binary finite-length machine words (32 b. or 64 b.).

⇒ Loss in accuracy, two consequences:
   1. Constant values are slightly altered.
   2. Rounding errors during computations.
Machine representation of real numbers

1. **Floating point: IEEE 754.**
   Not usual on microcontrollers.

   ![Floating point representation](image)

   
   number = sign × 2^{exponent + cst. offset} × fraction

   Correct rounding for base operations: +, -, *, /.

   ⇒ If [bounds on] operands are known, not special, far enough from extremal values, then rounding error is bounded for +, -, *, / (not /).

2. **Fixed point.**
   If operands are not special, far enough from extremal values, then rounding error is bounded for +, -, *.

3. **Two integers.**
Machine representation of real numbers

1. Floating point.
2. Fixed point.
   • Base behavior: +, −, *, / follow rational definition + fraction simplification:
     \[
     \frac{p_1}{q_1} + \frac{p_2}{q_2} = \text{simpl} \left( \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \right), \text{ etc.}
     \]
   No rounding error.
   Problem: numerator value can easily exceed integer bounds.
   • Approximated behavior to ensure bounded numerator.
Alteration of constants

With IEEE 754, 32 bits, constants

\[
\begin{align*}
Ac &= [0.4990, -0.0500; 0.0100, 1.0000]; \\
Bc &= [1;0]; \\
Cc &= [564.48, 0]; \\
Dc &= -1280;
\end{align*}
\]

become

\[
\begin{align*}
Ac &\approx [0.49900001287460327, -0.050000000074505806; \\
&\quad 0.009999999776482582, 1.0000]; \\
Bc &\approx [1;0]; \\
Cc &\approx [564.47998046875, 0]; \\
Dc &\approx -1280;
\end{align*}
\]
Effect on proof

\[ xc = \text{zeros}(2,1); \]
\[ xc \in \mathcal{E}_P \]
\[ \text{receive}(y, 2); \text{receive}(yd, 2); \]
\[ xc \in \mathcal{E}_P \]
\[ \text{while } 1 \]
\[ xc \in \mathcal{E}_P \]
\[ yc = \max(\min(y - yd, 1), -1); \]
\[ xc \in \mathcal{E}_P, \quad y_c^2 \leq 1 \]
\[ \begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q\mu} \quad \mid \quad Q_{\mu} = \begin{pmatrix} \mu P & 0_{2 \times 1} \\ 0_{1 \times 2} & 1 - \mu \end{pmatrix}, \mu = 0.9991 \]
\[ u = Cc*xc + Dc*yc; \]
\[ \begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q\mu} \]
\[ xc = Ac*xc + Bc*yc; \]
\[ xc \in \mathcal{E}_{\tilde{P}} \quad \mid \quad \tilde{P} = \left[ \begin{pmatrix} A_c & B_c \end{pmatrix} \cdot Q_{\mu}^{-1} \cdot \begin{pmatrix} A_c & B_c \end{pmatrix}^T \right]^{-1} \]
\[ \text{send}(u, 1); \]
\[ xc \in \mathcal{E}_{\tilde{P}} \]
\[ \text{receive}(y, 2); \]
\[ xc \in \mathcal{E}_{\tilde{P}} \]
\[ \text{receive}(yd, 2); \]
\[ xc \in \mathcal{E}_{\tilde{P}} \]
\[ xc \in \mathcal{E}_P \]
\[ \text{end} \]

Rest of the code and proof sketch unchanged.

\( \tilde{P} \) depends on \( A_c, B_c, C_c, D_c \), is altered.

\[ \Rightarrow \text{Last implication to be checked, might be wrong.} \]
Rounding errors

With real numbers, the implication

\[
\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_{\mu}} \\
x_c = A_c x_c + B_c y_c;
\]

\[
x_c \in \mathcal{E}_{\tilde{P}} \quad | \quad \tilde{P} = \left[ \begin{pmatrix} A_c & B_c \end{pmatrix} \cdot Q_{\mu}^{-1} \cdot \begin{pmatrix} A_c & B_c \end{pmatrix}^T \right]^{-1}
\]

holds.

With floats, + and * introduce rounding errors.

As \( x_c, y_c \) belong to an ellipsoid, they are bounded so the rounding error on \( x_c \) can be bounded by \( (e_1, e_2) \).
Super-ellipsoid

Let \( \tilde{E}_R \supset \tilde{E}_P \) an ellipse s.t.
\[
\forall x_c \in \tilde{E}_P, \ \forall x_c' \in \mathbb{R}^2, \ |x_{c_1}' - x_{c_1}| \leq e_1 \land |x_{c_2}' - x_{c_2}| \leq e_2 \implies x_c' \in \tilde{E}_R \quad (*)
\]

Then:
\[
\begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \tilde{E}_{Q_{\mu}} \\
xc = Ac*xc + Bc*y_c; \\
x_c \in \tilde{E}_R
\]

\( \tilde{E}_R \) can be the smallest magnification of \( \tilde{E}_P \) s.t. \( (*) \) holds.

Can be computed, whatever number of dimensions.
Effect on proof

\[ x_c = \text{zeros}(2,1); \]
\[ x_c \in \mathcal{E}_P \]
\[ \text{receive}(y,2); \text{receive}(yd,2); \]
\[ x_c \in \mathcal{E}_P \]
\[ \text{while } 1 \]
\[ x_c \in \mathcal{E}_P \]
\[ y_c = \max(\min(y - yd,1),-1); \]
\[ x_c \in \mathcal{E}_P, \quad y_c^2 \leq 1 \]
\[ \begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu} \quad | \quad Q_\mu = \begin{pmatrix} \mu P & 0_{2 \times 1} \\ 0_{1 \times 2} & 1 - \mu \end{pmatrix}, \mu = 0.9991 \]
\[ u = Cc*x_c + Dc*y_c; \]
\[ \begin{pmatrix} x_c \\ y_c \end{pmatrix} \in \mathcal{E}_{Q_\mu} \]
\[ x_c = Ac*x_c + Bc*y_c; \]
\[ x_c \in \mathcal{E}_{\tilde{R}} \]
\[ \text{send}(u,1); \]
\[ x_c \in \mathcal{E}_{\tilde{R}} \]
\[ \text{receive}(y,2); \]
\[ x_c \in \mathcal{E}_{\tilde{R}} \]
\[ \text{receive}(yd,2); \]
\[ x_c \in \mathcal{E}_{\tilde{R}} \]
\[ x_c \in \mathcal{E}_P \]
\[ \text{end} \]

Replace \( \mathcal{E}_P \) by \( \mathcal{E}_{\tilde{R}} \) in proof sketch.

Last implication to be checked, might be wrong.

Here it works: system stable with floats.
Other functions

Elementary operations +, * are sufficient for linear, invariant systems. The method applies if the proof sketch fits: no tight assumptions, complex operations on weakened invariants.

1-var, differentiable, periodic functions can be computed

- with an abacus and a polyhedral interpolation function
- with a polyhedral approximation

with a bounded error (\(\sin, \cos\)).

Idem for 1-var, differentiable functions restricted to a finite range. OK if proof ensures the operand is bounded to the range.
Proof checking on C code

Transformations from pseudocode to C code are not equivalences.  
⇒ The transposed proof sketch on C code might be false.  
⇒ Check the C code invariants with an analyzer.  

Attempt with PIPS.
Interrupt handlers

Specific aspect of the code with interrupt handler: initialization followed by main loop, that can be interrupted at any time by signals.

Problem: structures with parallel loops difficult to analyze.
Polyhedral analysis

PIPS performs **polyhedral analysis**: invariants = system of (in)equalities on the program variables (polyhedron). Good balance accuracy/complexity.

Usually, iterative approach: direct invariant propagation on control points, widening on cycles [Cousot-Halbwachs].

PIPS approach:

1. Abstract each instruction by a transfer relation (transformer), bottom to top. Links values before and after the instruction.

   \[
   x += y; \quad \leadsto \quad \{ x' = x + y \land y' = y \}
   \]

2. Invariant propagation on control points, using transformers.
Problem of parallel loops

When confronted to the code

```c
while (rand()) {
    if (rand()) {c_1}
    else {c_2}
}
```

PIPS computes transformers $T_1, T_2$ associated to codes $c_1, c_2$, then $T = T_1 \sqcup T_2$ the transformer of the whole loop body, then $T^*$ the transformer corresponding to the loop.

Problem: loss in accuracy with $\sqcup$ and $\ast$, amplified when combined. Too imprecise for many systems.
Different approaches

1. Refine transformers with invariants.
   Usual analysis with transformers then invariants.
   Then, restrict every transformer with its entry point invariant.
   Recompute invariants with new transformers.
   Does not converge in general.
   Rarely suited.

2. Delay convex hull.

3. Restructure the program.
Different approaches

1. Refine transformers with invariants.
2. Delay convex hull.
   Do not directly compute \( T = T_1 \sqcup T_2 \), then \( T^* \).
   Instead, keep track of the list \([T_1, T_2] \) of involved transformers.
   Later, to propagate invariant \( P \), do not compute
   \[
P' = T^*(P)
   \]
   but instead:
   \[
P' = \text{Comb}([T_1, T_2], P)
   \]
   with
   \[
   \text{Comb}([T_1, T_2], P) = P \sqcup T_1(P) \sqcup T_2(P) \sqcup T_1 \circ T_2(P) \sqcup T_2 \circ T_1(P) \\
   \sqcup T_1^+(P) \sqcup T_2^+(P) \sqcup T_1^+ \circ T_2 \circ T^*(P) \sqcup T_2^+ \circ T_1 \circ T^*(P)
   \]
3. Restructure the program.
Different approaches

1. Refine transformers with invariants.
2. Delay convex hull.
3. Restructure the program.

Transform into an equivalent program, easier to analyze. Idea: limit number of parallel loops by splitting control points according to loop guards.

Crucial point: choice of splitting partition. Manual or guided by a heuristic.
Different approaches

1. Refine transformers with invariants.
2. Delay convex hull.
3. Restructure the program.
   
   Best results: on 73 test cases, 
   28 → 63 with PIPS, 47 → 70 with ASPIC [Gonnord]. 
   Equivalence certified with Coq. 
   [NSAD’11].

Different approaches can be used simultaneously. 
Work in progress.
From physics to interrupt handlers: the real to float step

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November 20, 2012