# 15-213

***"The course that gives CMU its Zip!"***

# Integers
# Sep 3, 2002

## Topics

- **Numeric Encodings**
  - **Unsigned & Two's complement**
- **Programming Implications**
  - **C promotion rules**
- **Basic operations**
  - **Addition, negation, multiplication**
- **Programming Implications**
  - **Consequences of overflow**
  - **Using shifts to perform power-of-2 multiply/divide**

# C Puzzles

- Taken from old exams
- Assume machine with 32 bit word size, two's complement integers
- For each of the following C expressions, either:
  - Argue that is true for all argument values
  - Give example where not true

Initialization

```
int x = foo();

int y = bar();

unsigned ux = x;

unsigned uy = y;
```

```
• x < 0               ⇒  ((x*2) < 0)
• ux >= 0
• x & 7 == 7          ⇒  (x<<30) < 0
• ux > -1
• x > y               ⇒  -x < -y
• x * x >= 0
• x > 0 && y > 0      ⇒  x + y > 0
• x >= 0              ⇒  -x <= 0
• x <= 0              ⇒  -x >= 0
```

# Encoding Integers

## Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

## Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

**Sign Bit**

```
short int x =  15213;
short int y = -15213;
```

- **C `short` 2 bytes long**

| | Decimal | Hex | Binary |
|---|---|---|---|
| x | 15213 | 3B 6D | 00111011 01101101 |
| y | -15213 | C4 93 | 11000100 10010011 |

## Sign Bit

- **For 2's complement, most significant bit indicates sign**
  - **0 for nonnegative**
  - **1 for negative**

# Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

| Weight | 15213 | | -15213 | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 0 | 1 | 2 |
| 4 | 1 | 4 | 0 | 0 |
| 8 | 1 | 8 | 0 | 0 |
| 16 | 0 | 0 | 1 | 16 |
| 32 | 1 | 32 | 0 | 0 |
| 64 | 1 | 64 | 0 | 0 |
| 128 | 0 | 0 | 1 | 128 |
| 256 | 1 | 256 | 0 | 0 |
| 512 | 1 | 512 | 0 | 0 |
| 1024 | 0 | 0 | 1 | 1024 |
| 2048 | 1 | 2048 | 0 | 0 |
| 4096 | 1 | 4096 | 0 | 0 |
| 8192 | 1 | 8192 | 0 | 0 |
| 16384 | 0 | 0 | 1 | 16384 |
| -32768 | 0 | 0 | 1 | -32768 |
| **Sum** | | **15213** | | **-15213** |

# Numeric Ranges

## Unsigned Values

- *UMin* = 0
  **000…0**
- *UMax* = $2^w - 1$
  **111…1**

## Two's Complement Values

- *TMin* = $-2^{w-1}$
  **100…0**
- *TMax* = $2^{w-1} - 1$
  **011…1**

## Other Values

- Minus 1
  **111…1**

**Values for *W* = 16**

| | Decimal | Hex | Binary |
|---|---|---|---|
| `UMax` | 65535 | `FF FF` | `11111111 11111111` |
| `TMax` | 32767 | `7F FF` | `01111111 11111111` |
| `TMin` | -32768 | `80 00` | `10000000 00000000` |
| `-1` | -1 | `FF FF` | `11111111 11111111` |
| `0` | 0 | `00 00` | `00000000 00000000` |

# Values for Different Word Sizes

| | W | | | |
|---|---|---|---|---|
| | 8 | 16 | 32 | 64 |
| UMax | 255 | 65,535 | 4,294,967,295 | 18,446,744,073,709,551,615 |
| TMax | 127 | 32,767 | 2,147,483,647 | 9,223,372,036,854,775,807 |
| TMin | -128 | -32,768 | -2,147,483,648 | -9,223,372,036,854,775,808 |

## Observations

- $|TMin| = TMax + 1$
  - Asymmetric range
- $UMax = 2 * TMax + 1$

## C Programming

- `#include <limits.h>`
  - K&R App. B11
- Declares constants, e.g.,
  - `ULONG_MAX`
  - `LONG_MAX`
  - `LONG_MIN`
- Values platform-specific

# Unsigned & Signed Numeric Values

| *X* | B2U(*X*) | B2T(*X*) |
|---|---|---|
| `0000` | 0 | 0 |
| `0001` | 1 | 1 |
| `0010` | 2 | 2 |
| `0011` | 3 | 3 |
| `0100` | 4 | 4 |
| `0101` | 5 | 5 |
| `0110` | 6 | 6 |
| `0111` | 7 | 7 |
| `1000` | 8 | –8 |
| `1001` | 9 | –7 |
| `1010` | 10 | –6 |
| `1011` | 11 | –5 |
| `1100` | 12 | –4 |
| `1101` | 13 | –3 |
| `1110` | 14 | –2 |
| `1111` | 15 | –1 |

## Equivalence

- **Same encodings for nonnegative values**

## Uniqueness

- **Every bit pattern represents unique integer value**
- **Each representable integer has unique bit encoding**

## ⇒ Can Invert Mappings

- **U2B($x$) = B2U$^{-1}$($x$)**
  - **Bit pattern for unsigned integer**
- **T2B($x$) = B2T$^{-1}$($x$)**
  - **Bit pattern for two's comp integer**

# Casting Signed to Unsigned

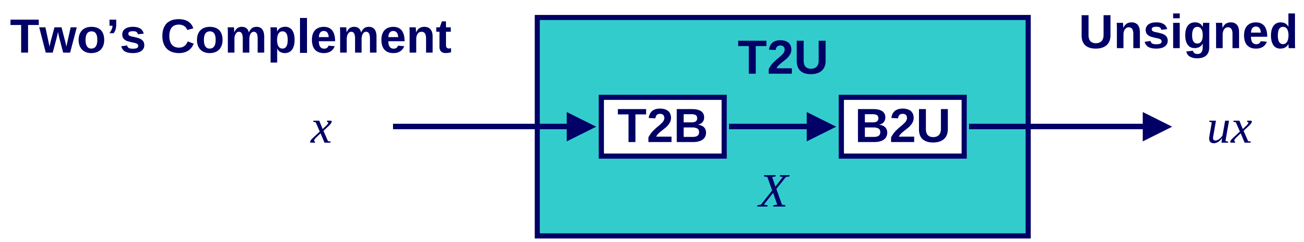
## C Allows Conversions from Signed to Unsigned

```
short int          x =  15213;
unsigned short int ux = (unsigned short) x;
short int          y  = -15213;
unsigned short int uy = (unsigned short) y;
```

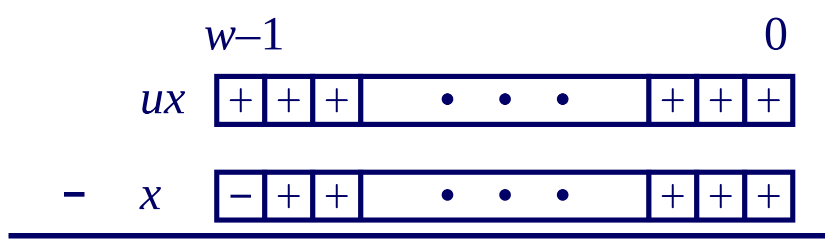
## Resulting Value

- No change in bit representation
- Nonnegative values unchanged
  - *ux* = 15213
- Negative values change into (large) positive values
  - *uy* = 50323

# Relation between Signed & Unsigned




**Maintain Same Bit Pattern**




$+2^{w-1} - (-2^{w-1}) = 2 \cdot 2^{w-1} = 2^w$

$$ux = \begin{cases} x & x \geq 0 \\ x + 2^w & x < 0 \end{cases}$$

# Relation Between Signed & Unsigned

| Weight | -15213 | | 50323 | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 1 | 2 |
| 4 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 |
| 16 | 1 | 16 | 1 | 16 |
| 32 | 0 | 0 | 0 | 0 |
| 64 | 0 | 0 | 0 | 0 |
| 128 | 1 | 128 | 1 | 128 |
| 256 | 0 | 0 | 0 | 0 |
| 512 | 0 | 0 | 0 | 0 |
| 1024 | 1 | 1024 | 1 | 1024 |
| 2048 | 0 | 0 | 0 | 0 |
| 4096 | 0 | 0 | 0 | 0 |
| 8192 | 0 | 0 | 0 | 0 |
| 16384 | 1 | 16384 | 1 | 16384 |
| 32768 | 1 | *-32768* | 1 | *32768* |
| **Sum** | | **-15213** | | **50323** |

- *uy* = *y* + 2 * 32768 = *y* + 65536

# Signed vs. Unsigned in C

## Constants

- By default are considered to be signed integers
- Unsigned if have "U" as suffix

```
0U, 4294967259U
```

## Casting

- Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

- Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```

# Casting Surprises

## Expression Evaluation

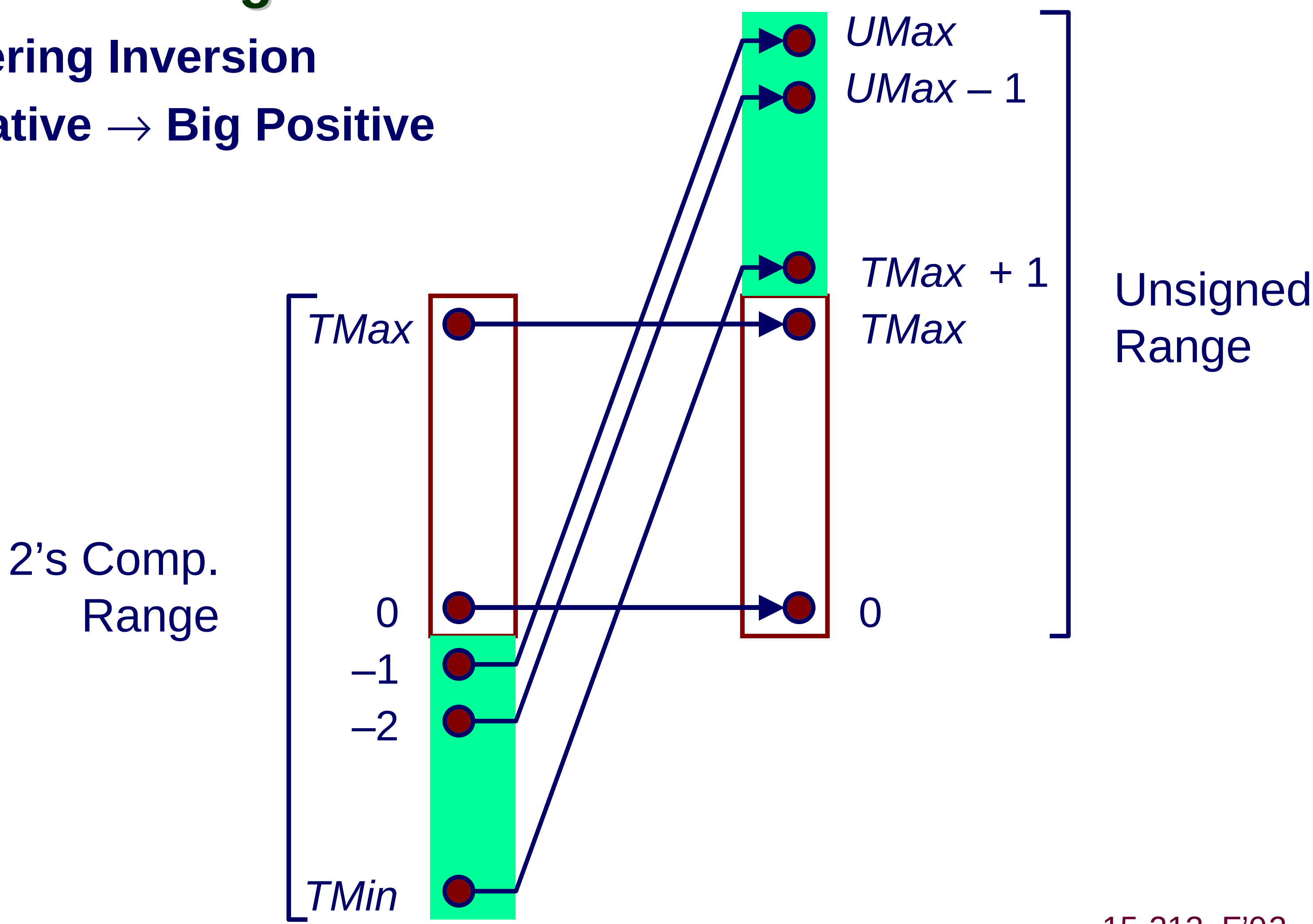
- If mix unsigned and signed in single expression, signed values implicitly cast to unsigned
- Including comparison operations <, >, ==, <=, >=
- Examples for *W* = 32

| Constant$_1$ | Constant$_2$ | Relation | Evaluation |
|---|---|---|---|
| `0` | `0U` | == | unsigned |
| `-1` | `0` | < | signed |
| `-1` | `0U` | > | unsigned |
| `2147483647` | `-2147483648` | > | signed |
| `2147483647U` | `-2147483648` | < | unsigned |
| `-1` | `-2` | > | signed |
| `(unsigned) -1` | `-2` | > | unsigned |
| `2147483647` | `2147483648U` | < | unsigned |
| `2147483647` | `(int) 2147483648U` | > | signed |

# Explanation of Casting Surprises

## 2's Comp. → Unsigned

- **Ordering Inversion**
- **Negative → Big Positive**



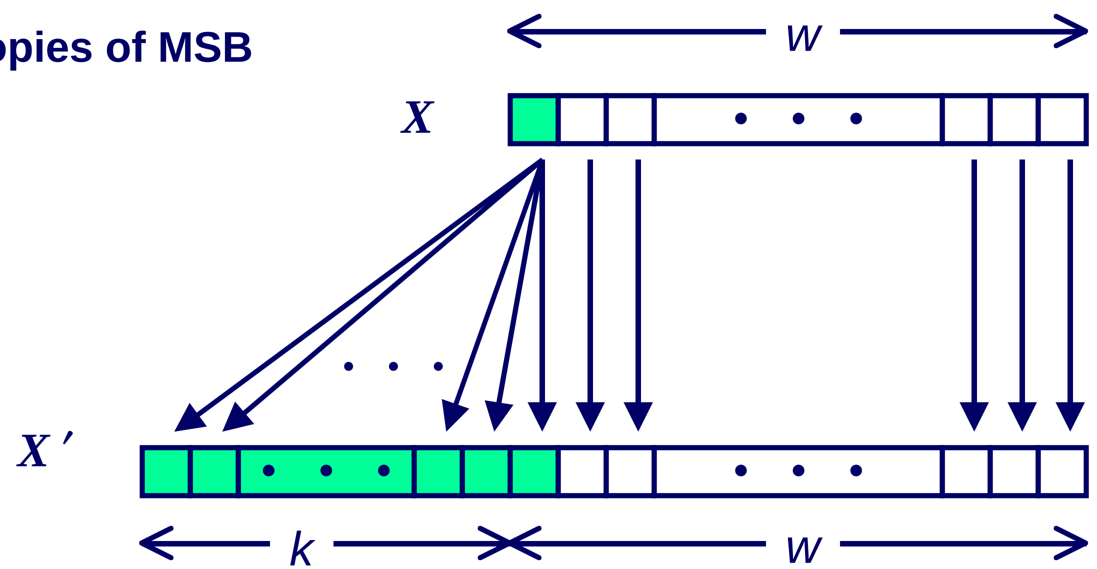
# Sign Extension

## Task:

- Given *w*-bit signed integer *x*
- Convert it to *w*+*k*-bit integer with same value

## Rule:

- Make *k* copies of sign bit:
- $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



# Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

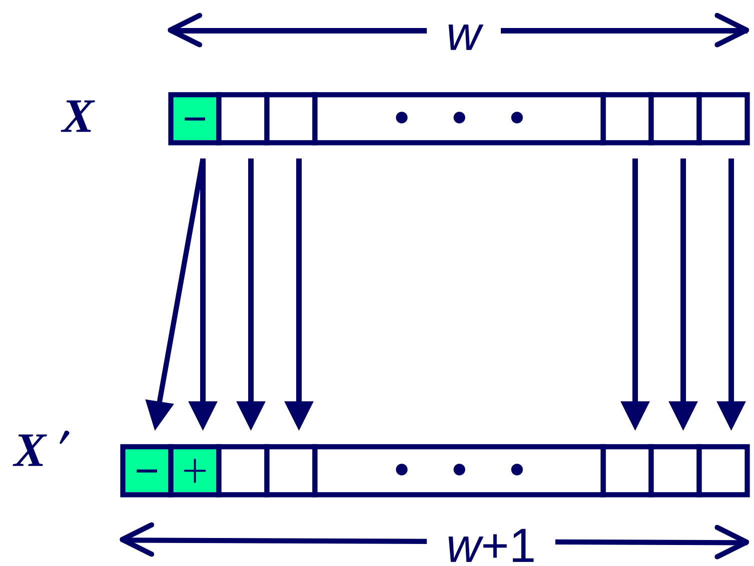
| | Decimal | Hex | Binary |
|---|---|---|---|
| x | 15213 | 3B 6D | 00111011 01101101 |
| ix | 15213 | 00 00 3B 6D | 00000000 00000000 00111011 01101101 |
| y | -15213 | C4 93 | 11000100 10010011 |
| iy | -15213 | FF FF C4 93 | 11111111 11111111 11000100 10010011 |

- Converting from smaller to larger integer data type
- C automatically performs sign extension

# Justification For Sign Extension

## Prove Correctness by Induction on *k*

- **Induction Step: extending by single bit maintains value**




- **Key observation:** $-2^{w-1} = -2^{w} + 2^{w-1}$
- **Look at weight of upper bits:**

$X \quad -2^{w-1}\, x_{w-1}$

$X' \quad -2^{w}\, x_{w-1} + 2^{w-1}\, x_{w-1} \quad = \quad -2^{w-1}\, x_{w-1}$

# Why Should I Use Unsigned?

## *Don't* Use Just Because Number Nonzero

- C compilers on some machines generate less efficient code

```
unsigned i;
for (i = 1; i < cnt; i++)
  a[i] += a[i-1];
```

- Easy to make mistakes

```
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

## *Do* Use When Performing Modular Arithmetic

- Multiprecision arithmetic
- Other esoteric stuff

## *Do* Use When Need Extra Bit's Worth of Range

- Working right up to limit of word size

# Negating with Complement & Increment

## Claim: Following Holds for 2's Complement

`~x + 1 == -x`

## Complement

- Observation: `~x + x == 1111…11`$_2$ `== -1`

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| `x` | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| `+ ~x` | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| `-1` | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

## Increment

- $\sim x + \cancel{x} + (-\cancel{x} + 1) \quad == \quad -\cancel{1} + (-x + \cancel{1})$
- `~x + 1        ==    -x`

## Warning: Be cautious treating `int`'s as integers

- OK here

# Comp. & Incr. Examples

x = 15213

| | Decimal | Hex | Binary |
|---|---|---|---|
| `x` | 15213 | `3B 6D` | `00111011 01101101` |
| `~x` | -15214 | `C4 92` | `11000100 10010010` |
| `~x+1` | -15213 | `C4 93` | `11000100 1001001`**`1`** |
| `y` | -15213 | `C4 93` | `11000100 10010011` |

0

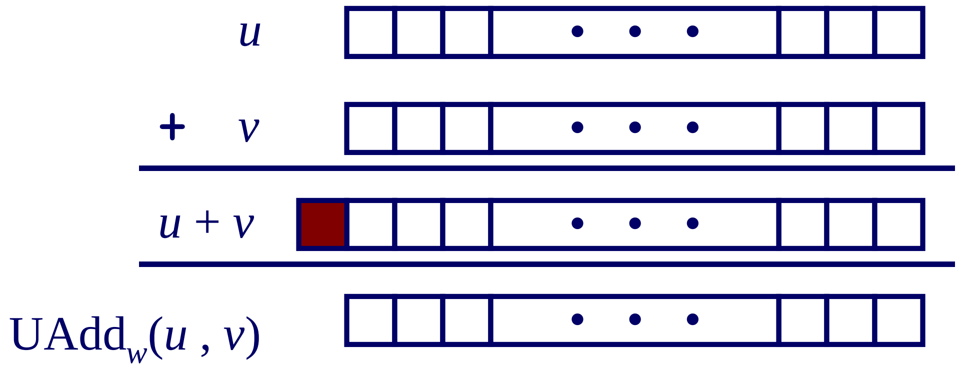
| | Decimal | Hex | Binary |
|---|---|---|---|
| `0` | 0 | `00 00` | `00000000 00000000` |
| `~0` | -1 | `FF FF` | `11111111 11111111` |
| `~0+1` | 0 | `00 00` | `00000000 00000000` |

# Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits




## Standard Addition Function

- **Ignores carry output**

## Implements Modular Arithmetic

$s \;=\; \mathrm{UAdd}_w(u\ ,\ v) \;=\; u + v \mod 2^w$

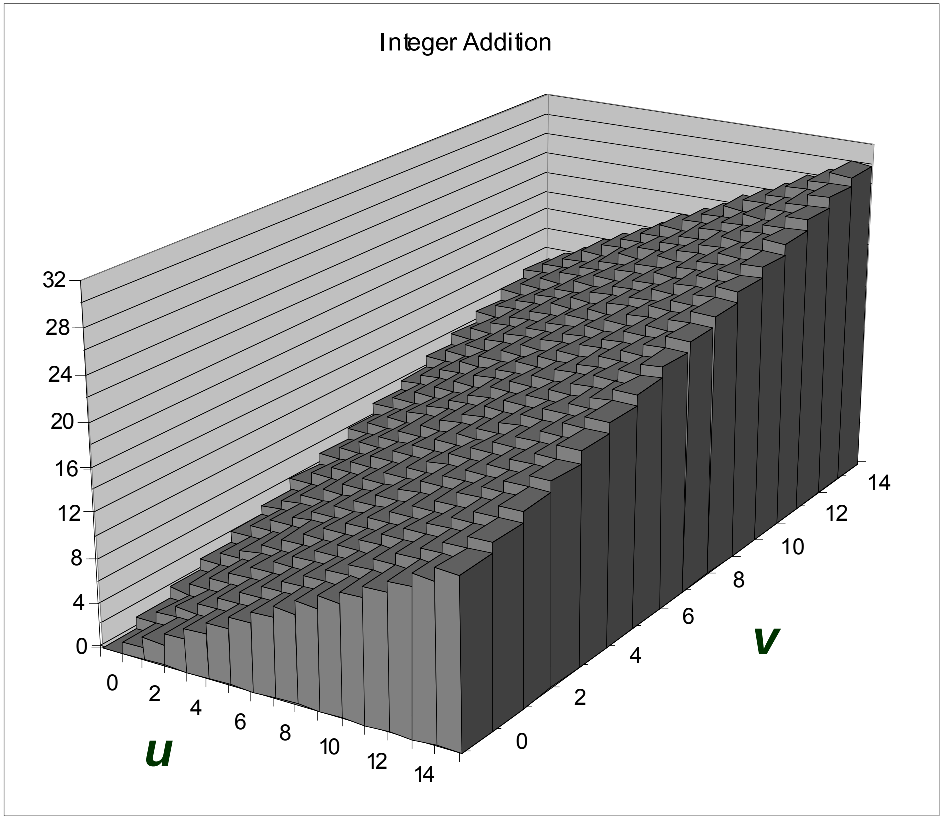
$$UAdd_w(u,v) \;=\; \begin{cases} u+v & u+v < 2^w \\ u+v-2^w & u+v \geq 2^w \end{cases}$$

# Visualizing Integer Addition

## Integer Addition

- 4-bit integers *u*, *v*
- Compute true sum $\text{Add}_4(u, v)$
- Values increase linearly with *u* and *v*
- Forms planar surface

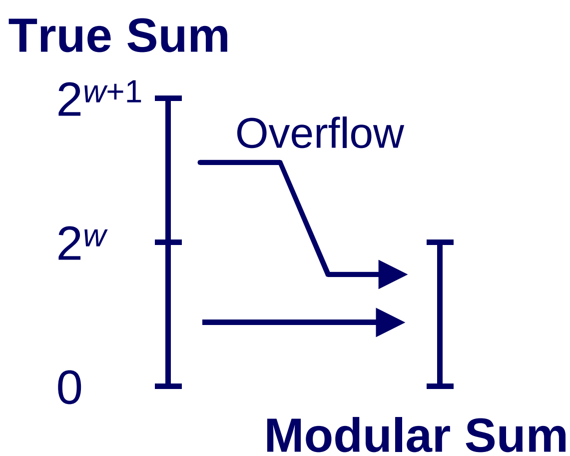
**$\text{Add}_4(u, v)$**

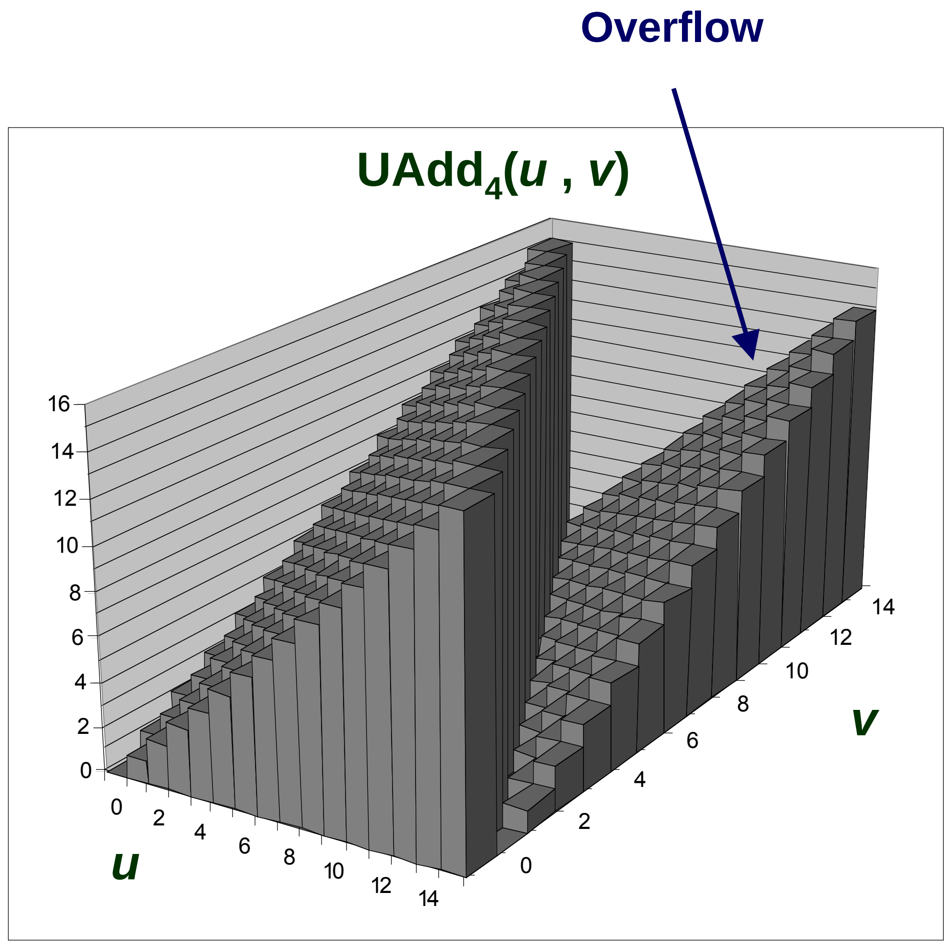


# Visualizing Unsigned Addition

## Wraps Around

- If true sum $\geq 2^w$
- At most once



# Mathematical Properties

## Modular Addition Forms an *Abelian Group*

- Closed under addition

  $0 \leq \mathrm{UAdd}_w(u, v) \leq 2^w - 1$

- Commutative

  $\mathrm{UAdd}_w(u, v) = \mathrm{UAdd}_w(v, u)$

- Associative

  $\mathrm{UAdd}_w(t, \mathrm{UAdd}_w(u, v)) = \mathrm{UAdd}_w(\mathrm{UAdd}_w(t, u), v)$

- 0 is additive identity

  $\mathrm{UAdd}_w(u, 0) = u$

- Every element has additive inverse
  - Let $\mathrm{UComp}_w(u) = 2^w - u$

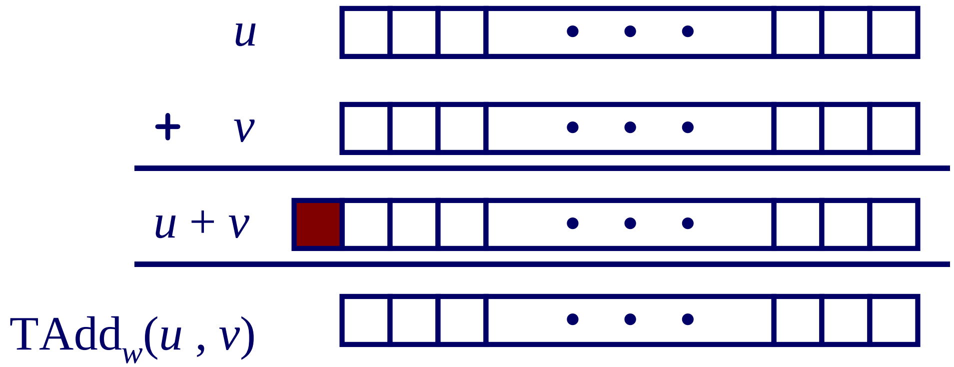
  $\mathrm{UAdd}_w(u, \mathrm{UComp}_w(u)) = 0$

# Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits




## TAdd and UAdd have Identical Bit-Level Behavior

- **Signed vs. unsigned addition in C:**

```
int s, t, u, v;
s = (int) ((unsigned) u + (unsigned) v);
 t = u + v
```

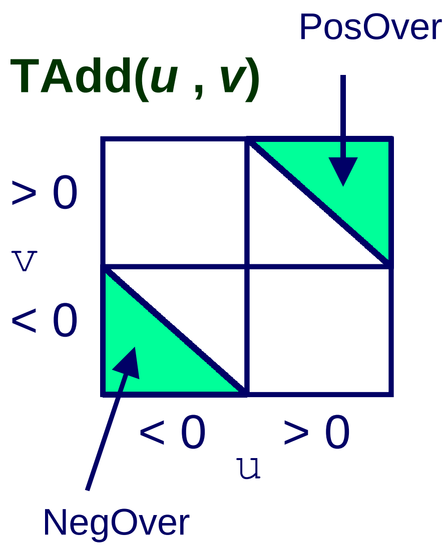
- **Will give** `s == t`

# Characterizing TAdd

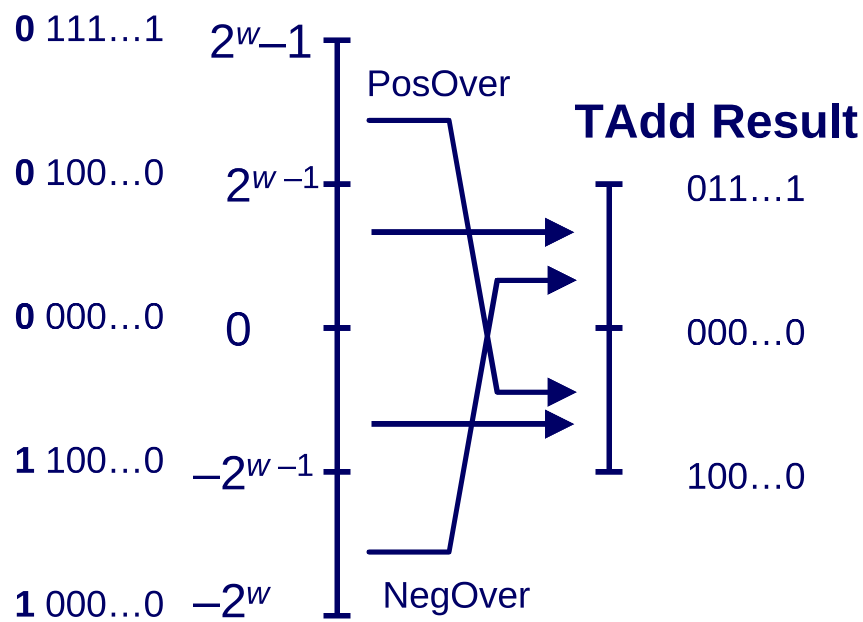
## Functionality

- True sum requires $w$+1 bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer

TAdd($u$ , $v$)




True Sum




$$TAdd_w(u,v) = \begin{cases} u+v+2^{w-1} & u+v < TMin_w \quad \textbf{(NegOver)} \\ u+v & TMin_w \le u+v \le TMax_w \\ u+v-2^{w-1} & TMax_w < u+v \quad \textbf{(PosOver)} \end{cases}$$

# Visualizing 2's Comp. Addition

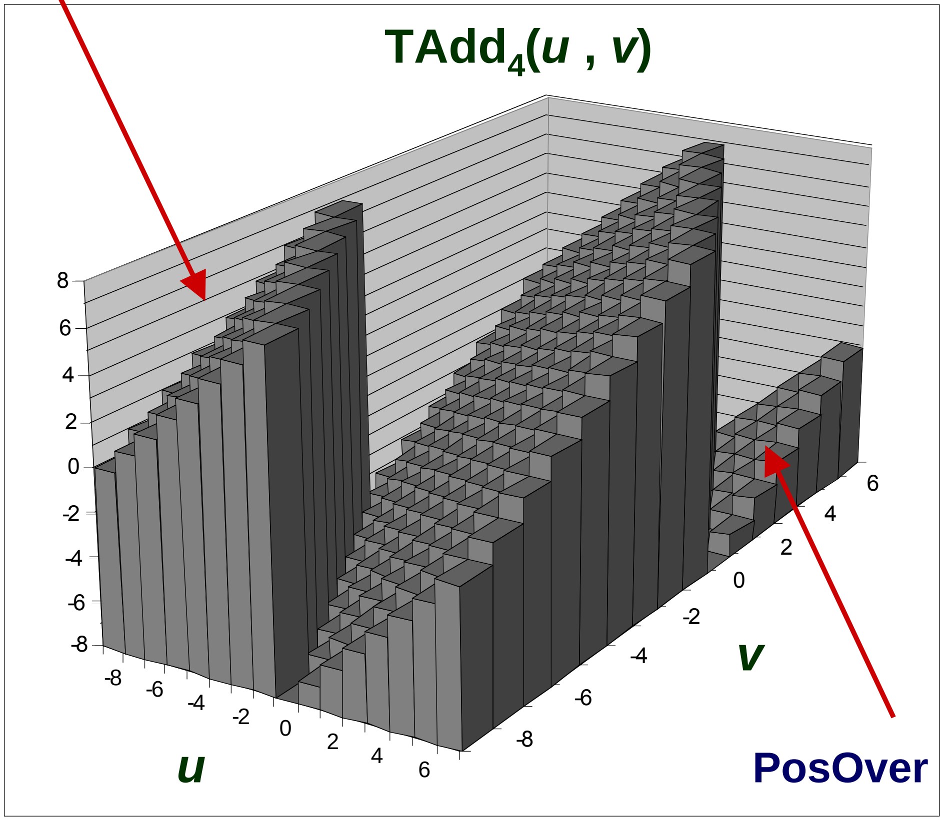
## Values

- 4-bit two's comp.
- Range from -8 to +7

## Wraps Around

- If sum $\geq 2^{w-1}$
  - Becomes negative
  - At most once
- If sum $< -2^{w-1}$
  - Becomes positive
  - At most once

NegOver



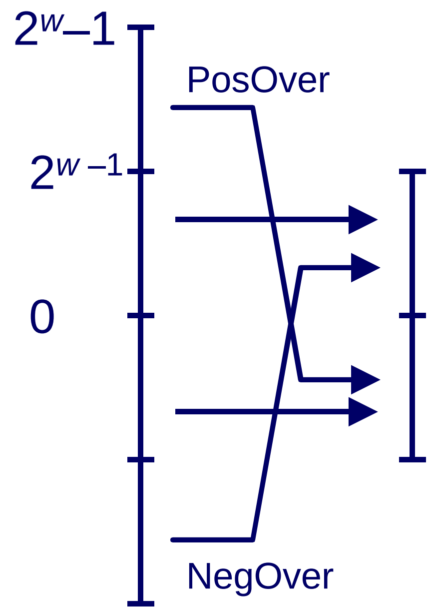

PosOver

# Detecting 2's Comp. Overflow

## Task

- **Given** $s$ = TAdd$_w$($u$ , $v$)
- **Determine if** $s$ = Add$_w$($u$ , $v$)
- **Example**

```
int s, u, v;
s = u + v;
```




## Claim

- **Overflow iff either:**

  ***u*, *v* < 0, *s* ≥ 0   (NegOver)**

  ***u*, *v* ≥ 0, *s* < 0   (PosOver)**

```
ovf = (u<0 == v<0) && (u<0 != s<0);
```

# Mathematical Properties of TAdd

## Isomorphic Algebra to UAdd

- $\text{TAdd}_w(u\ , v) = \text{U2T}(\text{UAdd}_w(\text{T2U}(u\ ), \text{T2U}(v)))$
  - **Since both have identical bit patterns**

## Two's Complement Under TAdd Forms a Group

- **Closed, Commutative, Associative, 0 is additive identity**
- **Every element has additive inverse**

  **Let** $\quad \textbf{TComp}_w\,(u\ ) = \textbf{U2T}(\textbf{UComp}_w(\textbf{T2U}(u\ ))$

  $\textbf{TAdd}_w(u\ , \textbf{TComp}_w\,(u\ )) = 0$

$$TComp_w(u) \quad = \quad \begin{cases} -u & u \neq TMin_w \\ TMin_w & u = TMin_w \end{cases}$$

# Multiplication

## Computing Exact Product of *w*-bit numbers *x*, *y*

- **Either signed or unsigned**

## Ranges

- **Unsigned:** $0 \le x * y \le (2^{w} - 1)^{2} = 2^{2w} - 2^{w+1} + 1$
  - **Up to 2*w* bits**
- **Two's complement min:** $x * y \ge (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
  - **Up to 2*w*–1 bits**
- **Two's complement max:** $x * y \le (-2^{w-1})^{2} = 2^{2w-2}$
  - **Up to 2*w* bits, but only for $(TMin_w)^2$**

## Maintaining Exact Results

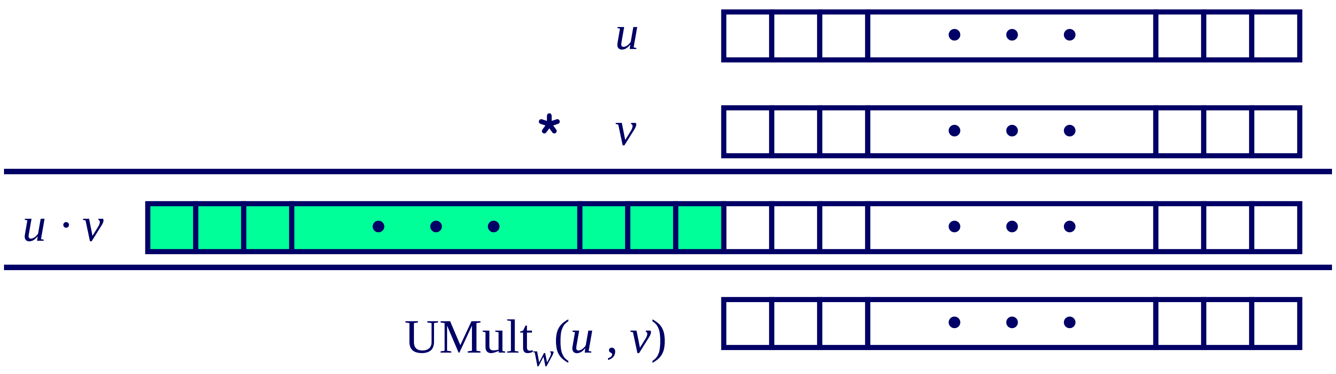
- **Would need to keep expanding word size with each product computed**
- **Done in software by "arbitrary precision" arithmetic packages**

# Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

Discard *w* bits: *w* bits




## Standard Multiplication Function

- **Ignores high order *w* bits**

## Implements Modular Arithmetic

$\text{UMult}_w(u, v) \quad = \quad u \cdot v \bmod 2^w$

# Unsigned vs. Signed Multiplication

## Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

- **Truncates product to *w*-bit number** *up* **=** $\mathrm{UMult}_w(ux, uy)$
- **Modular arithmetic: *up* = *ux* · *uy* mod 2*^w^*** $up = ux \cdot uy \bmod 2^w$

## Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

- **Compute exact product of two *w*-bit numbers *x*, *y***
- **Truncate result to *w*-bit number** *p* = $\mathrm{TMult}_w(x, y)$

# Unsigned vs. Signed Multiplication

## Unsigned Multiplication

```
unsigned ux = (unsigned) x;
unsigned uy = (unsigned) y;
unsigned up = ux * uy
```

## Two's Complement Multiplication

```
int x, y;
int p = x * y;
```

## Relation

- Signed multiplication gives same bit-level result as unsigned
- `up == (unsigned) p`

# Power-of-2 Multiply with Shift

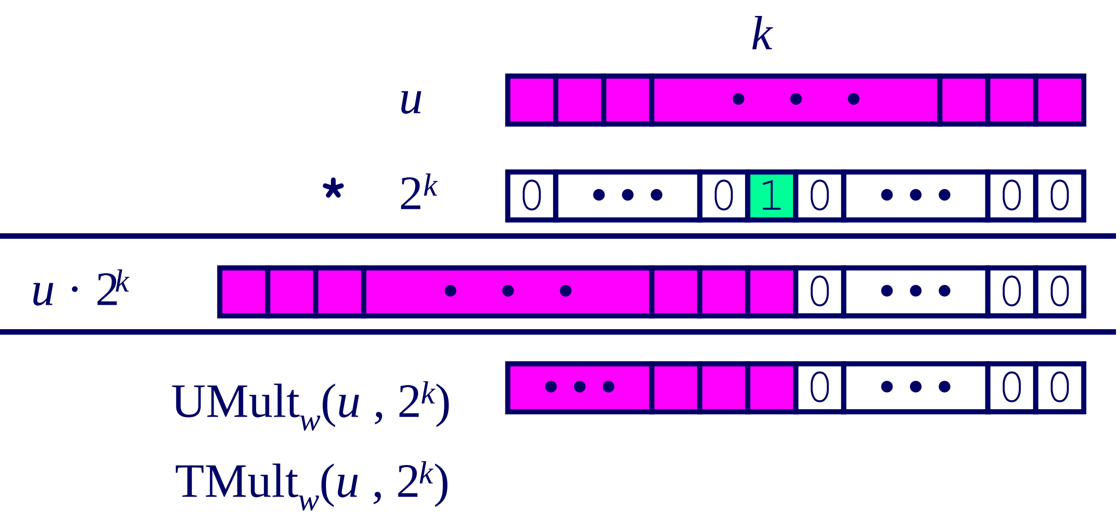
## Operation

- `u << k` gives `u *` $2^k$
- Both signed and unsigned

Operands: $w$ bits

True Product: $w+k$ bits

Discard $k$ bits: $w$ bits



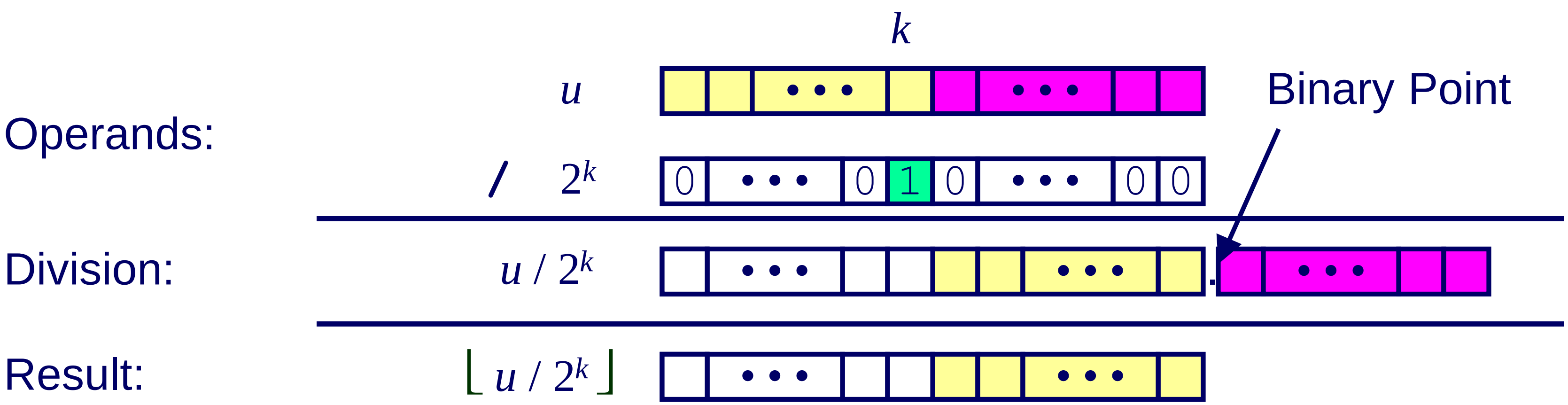

## Examples

- `u << 3          ==   u * 8`
- `u << 5 - u << 3      ==     u * 24`
- Most machines shift and add much faster than multiply
  - Compiler generates this code automatically

# Unsigned Power-of-2 Divide with Shift

## Quotient of Unsigned by Power of 2

- `u >> k` gives $\lfloor$ `u / ` $2^k \rfloor$
- Uses logical shift

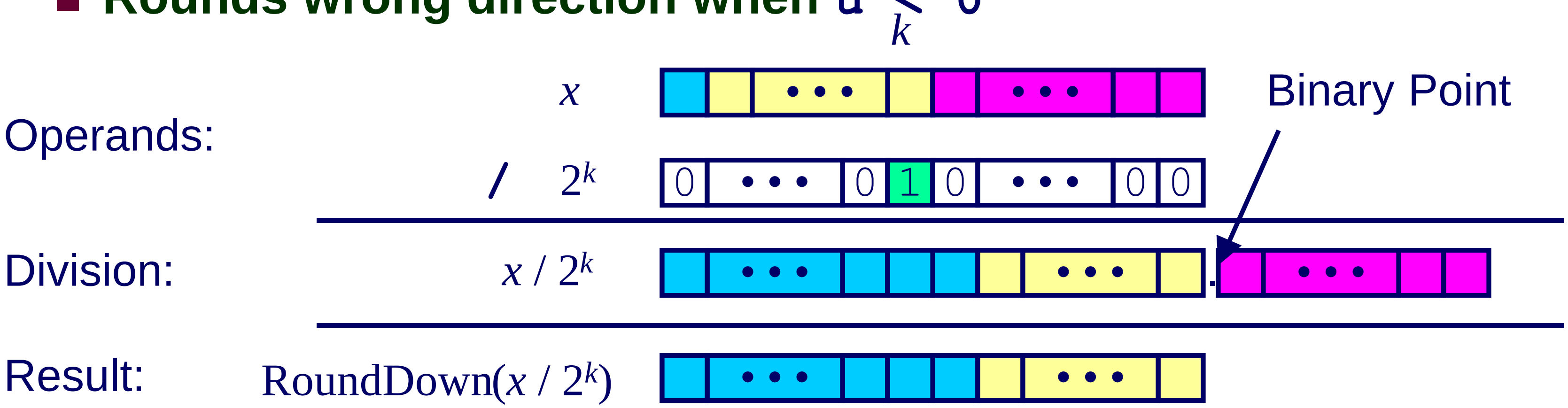



| | Division | Computed | Hex | Binary |
|---|---|---|---|---|
| `x` | 15213 | 15213 | `3B 6D` | `00111011 01101101` |
| `x >> 1` | 7606.5 | 7606 | `1D B6` | **0**`0011101 10110110` |
| `x >> 4` | 950.8125 | 950 | `03 B6` | **0000**`0011 10110110` |
| `x >> 8` | 59.4257813 | 59 | `00 3B` | **00000000** `00111011` |

# Signed Power-of-2 Divide with Shift

## Quotient of Signed by Power of 2

- **`x >> k` gives ⌊ `x / ` $2^k$ ⌋**
- **Uses arithmetic shift**
- **Rounds wrong direction when `u < 0`**



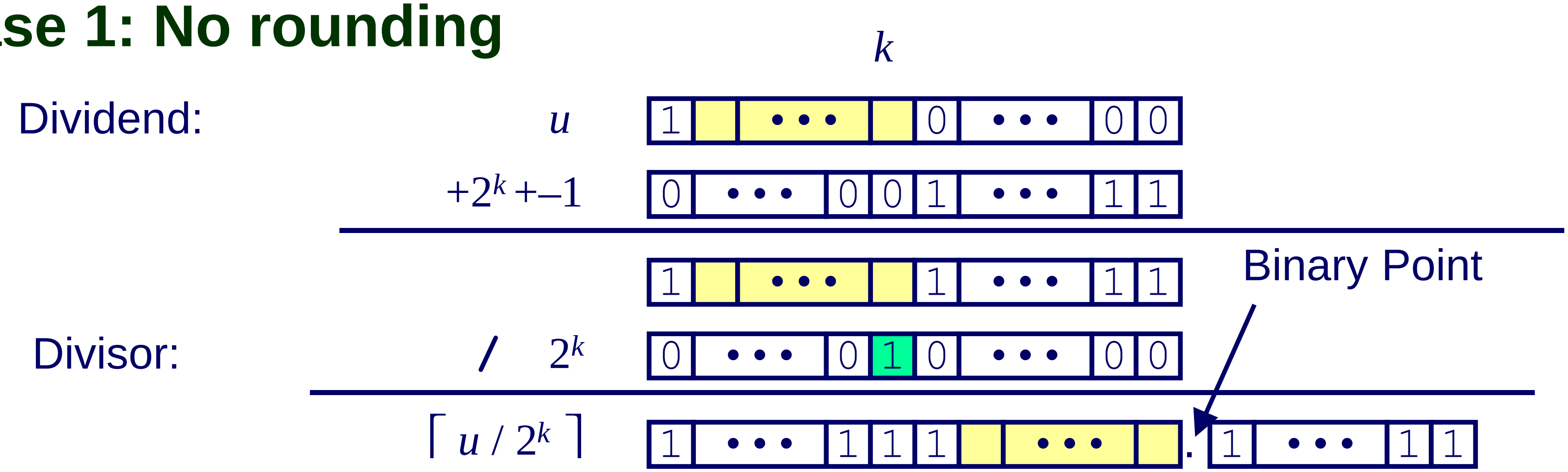

| | Division | Computed | Hex | Binary |
|---|---|---|---|---|
| `y` | -15213 | -15213 | `C4 93` | `11000100 10010011` |
| `y >> 1` | -7606.5 | -7607 | `E2 49` | **`1`**`1100010 01001001` |
| `y >> 4` | -950.8125 | -951 | `FC 49` | **`1111`**`1100 01001001` |
| `y >> 8` | -59.4257813 | -60 | `FF C4` | **`11111111`** `11000100` |

# Correct Power-of-2 Divide

## Quotient of Negative Number by Power of 2

- **Want $\lceil$ `x / 2`$^k$ $\rceil$ (Round Toward 0)**
- **Compute as $\lfloor$ `(x+2`$^k$`-1)/ 2`$^k$ $\rfloor$**
  - **In C: `(x + (1<<k)-1) >> k`**
  - **Biases dividend toward 0**

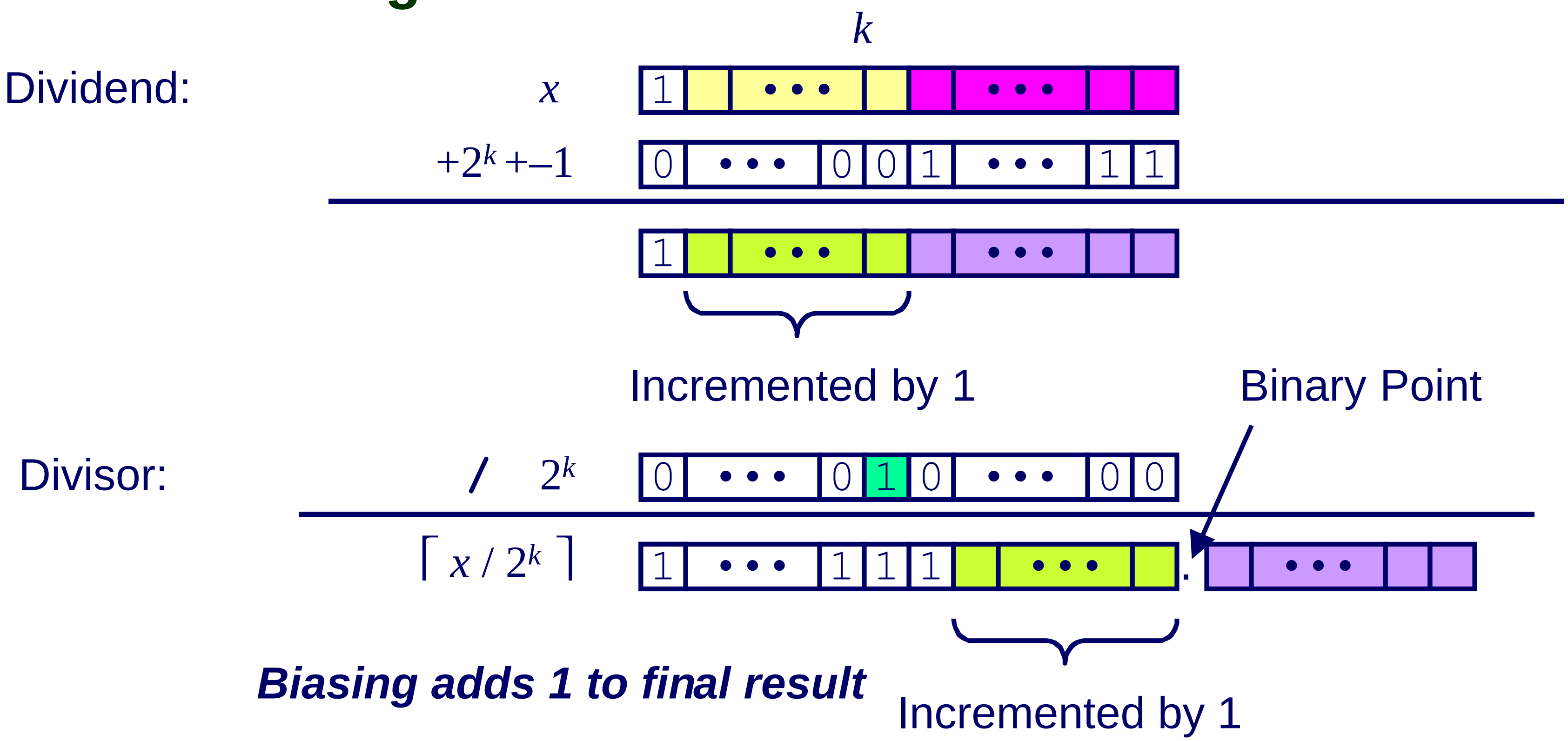
## Case 1: No rounding




***Biasing has no effect***

# Correct Power-of-2 Divide (Cont.)

## Case 2: Rounding




***Biasing adds 1 to final result***

# Properties of Unsigned Arithmetic

## Unsigned Multiplication with Addition Forms Commutative Ring

- Addition is commutative group
- Closed under multiplication

  $0 \leq \text{UMult}_w(u\ ,\ v) \leq 2^w - 1$
- Multiplication Commutative

  $\text{UMult}_w(u\ ,\ v) = \text{UMult}_w(v\ ,\ u)$
- Multiplication is Associative

  $\text{UMult}_w(t,\ \text{UMult}_w(u\ ,\ v)) = \text{UMult}_w(\text{UMult}_w(t,\ u\ ),\ v)$
- 1 is multiplicative identity

  $\text{UMult}_w(u\ ,\ 1) = u$
- Multiplication distributes over addtion

  $\text{UMult}_w(t,\ \text{UAdd}_w(u\ ,\ v)) = \text{UAdd}_w(\text{UMult}_w(t,\ u\ ),\ \text{UMult}_w(t,\ v))$

# Properties of Two's Comp. Arithmetic

## Isomorphic Algebras

- Unsigned multiplication and addition
  - Truncating to *w* bits
- Two's complement multiplication and addition
  - Truncating to *w* bits

## Both Form Rings

- Isomorphic to ring of integers mod $2^w$

## Comparison to Integer Arithmetic

- Both are rings
- Integers obey ordering properties, e.g.,

  $u > 0 \quad \Rightarrow \quad u + v > v$

  $u > 0, v > 0 \quad \Rightarrow \quad u \cdot v > 0$

- These properties are not obeyed by two's comp. arithmetic

  *TMax* `+ 1` `==` *TMin*

  `15213 * 30426` `==` `-10030` (16-bit words)

# C Puzzle Answers

- Assume machine with 32 bit word size, two's comp. integers
- *TMin* makes a good counterexample in many cases

| | | | | |
|---|---|---|---|---|
| `x < 0` | $\Rightarrow$ | `((x*2) < 0)` | **False:** | *TMin* |
| `ux >= 0` | | | **True:** | 0 = *UMin* |
| `x & 7 == 7` | $\Rightarrow$ | `(x<<30) < 0` | **True:** | $x_1 = 1$ |
| `ux > -1` | | | **False:** | 0 |
| `x > y` | $\Rightarrow$ | `-x < -y` | **False:** | `-1`, *TMin* |
| `x * x >= 0` | | | **False:** | `30426` |
| `x > 0 && y > 0` | $\Rightarrow$ | `x + y > 0` | **False:** | *TMax*, *TMax* |
| `x >= 0` | $\Rightarrow$ | `-x <= 0` | **True:** | –*TMax* < 0 |
| `x <= 0` | $\Rightarrow$ | `-x >= 0` | **False:** | *TMin* |