Natural Algorithms and Influence Systems

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March 7th 2014

The living world speaks the language of algorithms

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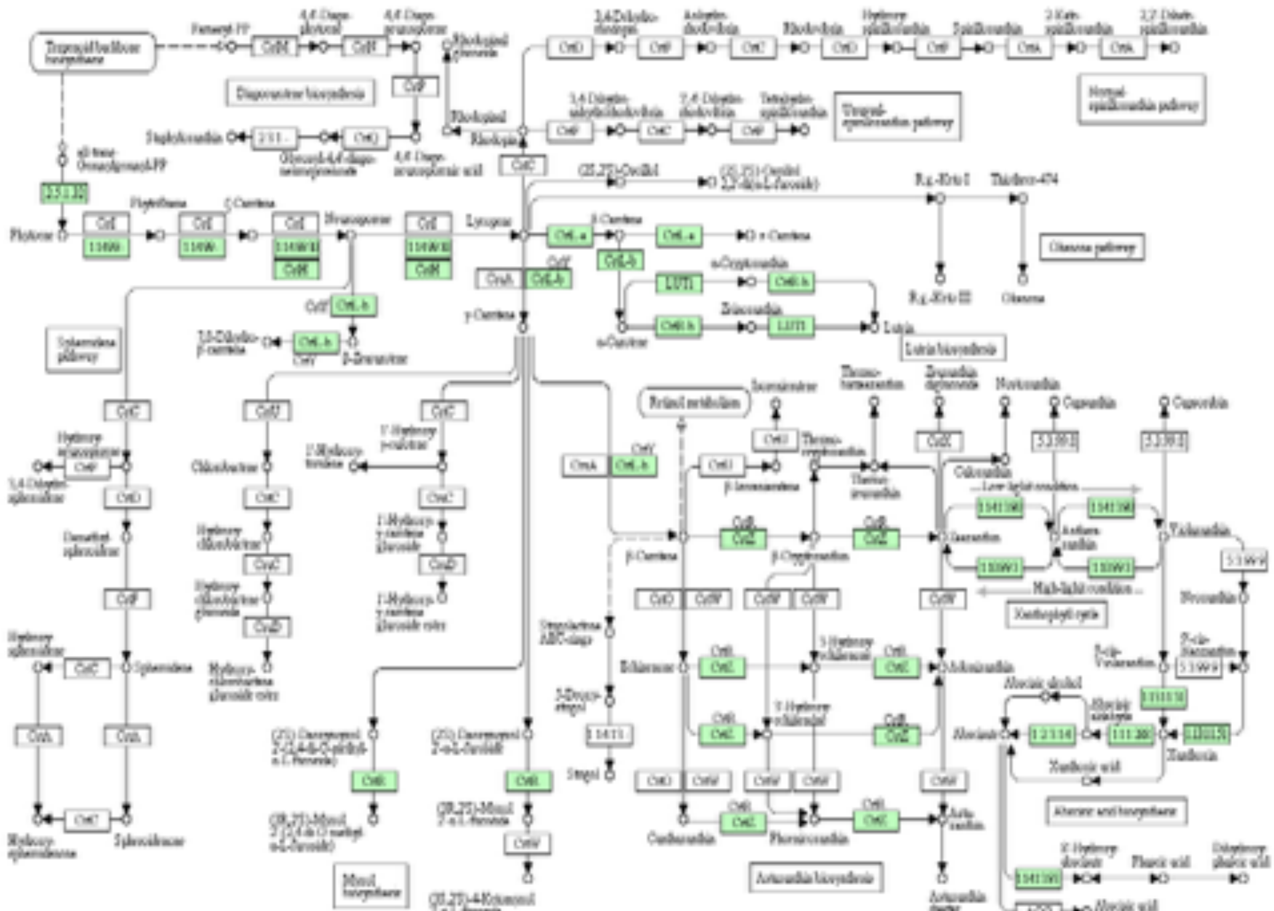
Why the sky is blue ?

vs.

Why tomatoes are red ?



PDE

Natural algorithm

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loops, conditionals, memory…

PDE

Natural algorithm

not human-designed

Part I : consensus algorithms

Part II : influence systems

Consensus in a multiagent system

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- The state of the agent i is captured by a variable x_i whose value at time t is denoted $x_i(t) \in \mathbb{R}^{d=1}$
- At time $t \geq 1$, each agent i updates x_i with a weighted average of of the values of its outgoing neighbors in the directed graph G_t.

Consensus in a multiagent system (cont'd)

$$x_i(t+1) = \sum_{j \in Out_i(G_t)} A_{i\,j}(t)\; x_j(t)$$

with $\sum_{j \in Out_i(G_t)} A_{i\,j}(t) = 1$ and $A_{i\,j}(t) > 0$

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where A is a stochastic matrix with entries in $\{0\} \cup [\alpha, 1]$

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$$\rightsquigarrow x_i(t+1) \in \big[(1-\alpha)m_i(t) + \alpha M_i(t)\,,\; \alpha m_i(t) + (1-\alpha)M_i(t)\big]$$

$$\text{with } \begin{cases} m_i(t) = \min\{x_j(t) : j \in Out_i(G_t)\} \\ M_i(t) = \max\{x_j(t) : j \in Out_i(G_t)\} \end{cases}$$

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$$\begin{cases} x(t+1) = P(t)\; x(0) \\ P(t) = A(t) \dots A(0) \end{cases}$$

Theorem : The sequence of (stochastic) matrices $\big(P(t)\big)_{t \geq 0}$ converges to a rank one (stochastic) matrix

Part I.a : Convergence and consensus

Assumptions

A1: Every matrix $A(t)$ is stochastic

A2: For each agent i and each time t, $A_{i\,i}(t) > 0$

A3: Non null entries of $A(t)$ are uniformly lower bounded by $\alpha > 0$

$$A_{i\,j}(t) \in \{0\} \cup [\alpha, 1]$$

Communication graphs

The graph associated to the matrix $A(t)$ is $G_t = ([N], E_t)$:

$$A_{ij}(t) > 0 \;\text{ iff }\; (i,j) \in E_t$$

Some results

- The Perron-Frobenius theorem

$$\forall t \in \mathbb{N},\ A(t) = A \text{ where } A \text{ is ergodic}$$

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$$\forall t \in \mathbb{N},\ A(t) \in \mathcal{M}$$

where $\mathcal{M}$ is finite and each finite product of matrices from $\mathcal{M}$ is ergodic

Some results

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- Bounded intercommunication intervals [Tsitsiklis 84]

$$\forall t \in \mathbb{N}, \quad A(t+\Phi)\cdots A(t) > 0$$

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$$\forall t \in \mathbb{N},\ G_t \text{ is oriented}$$

i.e., there exists some node j such that each node i is connected to j by a walk from i to j.

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- The Perron-Frobenius theorem
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- Bounded intercommunication intervals [Tsitsiklis 84]
- The coordinated model [Cao, Spielman, Morse 05]
- The bidirectional model [Moreau 05]
  1. $\forall t \in \mathbb{N},\ G_t$ is bidirectional
  2. $\forall s \in \mathbb{N},\ \cup_{t \geq s} G_t$ is strongly connected

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- Bounded intercommunication intervals [Tsitsiklis 84]
- The coordinated model [Cao, Spielman, Morse 05]
- The bidirectional model [Moreau 05]
- The decentralized model [Touri, Nedič 11]
  1. $\forall t \in \mathbb{N},\ G_t$ is semi-simple
  2. $\forall s \in \mathbb{N},\ \cup_{t \geq s} G_t$ est fortement connexe

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- The decentralized model [Touri, Nedìc 11]
  $\rightsquigarrow$ What happens if condition 2 does not hold?

Part I.b : Speed of convergence

Speed of convergence

- **Rate of convergence**

$$\rho = \sup_{x(0) \notin \mathbb{R}\mathbf{1} \ \wedge \ x(0) \in B} \Big(\lim_{t \to \infty} (\|x(t) - x^*\|)^{1/t} \Big)$$

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- **Convergence time**

$$T(\epsilon) = \inf\{\tau \ : \ \forall t \geq \tau, \forall x(o) \in B, \ \|x(t) - x^*\| \leq \epsilon\}$$

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Theorem [Olshevsky, Tsitsiklis 11] :

In the bidirectional Equal Neighbor model,

$1 - \frac{\gamma_1}{N^3} \leq \rho \leq 1 - \frac{\gamma_2}{N^3}$ et $T_N(\epsilon) = \Omega(N^3 \log\left(\frac{N}{\epsilon}\right))$

$\rightsquigarrow$ [C, Nowak 13] Extension to the general **constant** case

Time-varying influence

Two problematic issues with time-varying influence matrices:

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[Chazelle 11] : For each initial state $x(0) \in B$,

$$E(s) = \sum_{t \geq 0} \sum_{(i,j) \in E_t} |x_i(t) - x_j(t)|^s$$

Time-varying influence with bidirectional graphs

Theorem [Chazelle 11] : The total energy of a N-agent bidirectional system following the CH algorithm from an initial state $x(0) \in B$ satisfies

$$E(s) \leq \begin{cases} \alpha^{-O(N)} & \text{if } \; s = 1 \\ s^{1-N}\, \alpha^{-N^2+O(1)} & \text{if } \; 0 < s < 1 \end{cases}$$

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Corollary [Chazelle 11] : The number of ϵ-nontrivial steps of the CH algorithm from an initial state $x(0) \in B$ in a N-agent bidirectional system satisfies

$$T_N^*(\epsilon) \leq \frac{1}{N}\alpha^{1-N}$$

and this bound is tight

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$\rightsquigarrow$ An **algorithmic proof** for $s = 1$

Part II : Influence Systems

Influence Systems

- **communication algorithm**: each agent i determines its outgoing neighbors (= agents which inluence i)

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    - join agent i to six nearest neighbors

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    $\rightsquigarrow$ diffusive systems : each f_i is a linear convex combination of the x_k's

Hegselmann-Krause Systems

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Theorem [Blondel, Hendrickx, Tsitsiklis 09] : The HK system converges in finite time.

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$\rightsquigarrow$ **non diffusive** system

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[Vicsek, Cucker, Smale]

Theorem [Cucker, Smale 07], [Chazelle 09] : Conditions under which birds asymptotically adopt the same velocity

Influence Systems

Examples :

1. the HK model
2. Flocking
3. Chemiotaxis, the Ising model, neural networks, population dynamics, ...

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Theorem [Chazelle 13] : Diffusive systems are asymptotically periodic almost surely

Thanks !