Church-Rosser Properties of Normal Rewriting

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Outline

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1. Rewriting by examples
2. Confluence properties of plain rewriting
3. Normal rewriting
4. Conclusion

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3. Normal rewriting
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Plain Rewriting

$$\begin{array}{llcl} \text{Inv} & (x+y)^{-1} & \to & y^{-1}+x^{-1} \\ \text{Z} & x+0 & \to & x \\ \text{A} & (x+y)+z & \to & x+(y+z) \end{array}$$

Plain rewriting uses plain pattern matching

Rewriting Modulo

$$
\begin{aligned}
(x+y)^{-1} &\rightarrow y^{-1}+x^{-1} \\
0+x &\rightarrow x \\
(x+y)+z &= x+(y+z) \\
x+y &= y+x
\end{aligned}
$$

Rewriting modulo:

$$
(1+2)^{-1}+0 \rightarrow (2^{-1}+1^{-1})+0 \rightarrow 1^{-1}+2^{-1}
$$

Rewriting modulo uses pattern matching modulo equations

$$\begin{aligned}(x+y)^{-1} &\to y^{-1}+x^{-1}\\ 0+x &\hookrightarrow x\\ (x+y)+z &= x+(y+z)\\ x+y &= y+x\end{aligned}$$

Normal rewriting:

$$(1+2)^{-1}+0 \hookrightarrow (2+1)^{-1} \to 2^{-1}+1^{-1}$$

Uses normalization wrt simplifiers first and then pattern matching rules modulo all equations

Rewriting in λ-calculus

[Barendregt and Klop]:

$$
\begin{array}{rcl}
\omega\,1 & = & (\lambda x.(x\ x)\ \lambda s.\lambda z.(s\ z)) \\
& \longrightarrow & (\lambda s.\lambda z.(s\ z)\ \lambda s.\lambda z.(s\ z)) \\
& \longrightarrow & \lambda z.(\lambda s.\lambda z.(s\ z)\ z) \\
& \longrightarrow & \lambda z.\lambda z.(z\ z) \qquad -\mathit{wrong} \\
& \overset{(\geq 1)^*}{\underset{\alpha}{\longleftrightarrow}} & \lambda z.(\lambda s'.\lambda z'.(s'\ z')\ z) \\
& \overset{1}{\underset{\beta}{\longrightarrow}} & \lambda z.\lambda z'.(z\ z')
\end{array}
$$

β-reduction rewrites modulo α-conversion

Rewriting in λ-calculus

[Barendregt and Klop]:

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$$\begin{array}{rl} \overset{(\geq 1)^*}{\underset{\alpha}{\longleftrightarrow}} & \lambda z.(\lambda s'.\lambda z'.(s'\ z')\ z) \\ \overset{1}{\underset{\beta}{\longrightarrow}} & \lambda z.\lambda z'.(z\ z') \end{array}$$

β-reduction rewrites modulo α-conversion

Rewriting with recursors in Coq

$$\begin{array}{rcl} rec(0,u,f) & \to & u \\ rec(s(y),u,f) & \to & @(f,y,rec(y,u,f)) \\ @(\lambda z.u,v) & \to & u\{z \mapsto v\} \end{array}$$

rewrite:

$$rec(s(0),1,\lambda xy.+(x,y)) \to$$
$$@(\lambda xy.+(x,y),0,rec(0,1,\lambda xy.+(x,y))) \to$$
$$@(\lambda xy.+(x,y),0,1) \to +(0,1) \to 1$$

Uses plain pattern matching wrt constructors 0, S, and pattern matching modulo α for binders

Higher-order rewriting [Nipkow]

rules (differentiation):

$$\begin{aligned} \mathit{diff}(\lambda x.\mathit{sin}(f(x))) &\to \lambda x.\mathit{cos}(f(x)) * \mathit{diff}(f) \\ \mathit{diff}(\lambda x.x) &\to \lambda x.1 \end{aligned}$$

rewrite:

$$\begin{aligned} \mathit{diff}(\lambda x.\mathit{sin}(x)) &\overset{\Lambda}{\underset{\beta}{\longleftrightarrow}} \mathit{diff}(\lambda x.\mathit{sin}(\lambda x.x\ x)) \\ &\longrightarrow \lambda x.\mathit{cos}(x) * \mathit{diff}(\lambda x.x) \\ &\longrightarrow \lambda x.\mathit{cos}(x) * \mathit{diff}(\lambda x.x) \\ &\longrightarrow \lambda x.\mathit{cos}(x) * \lambda x.1 \\ &\longrightarrow \lambda x.\mathit{cos}(x) \end{aligned}$$

Higher-order rewriting is an instance of normal rewriting modulo beta, eta and alpha.

Questions

1. What is a general definition of rewriting ?
2. is my rewriting calculus terminating ?
3. is my rewriting calculus confluent ?

We focus on:

- Definition of normal rewriting
- Confluence assuming termination
- General abstract results
- Application to higher-order rewriting
- A treatment of binders as a particular case
- Flexibility of higher-order definitions

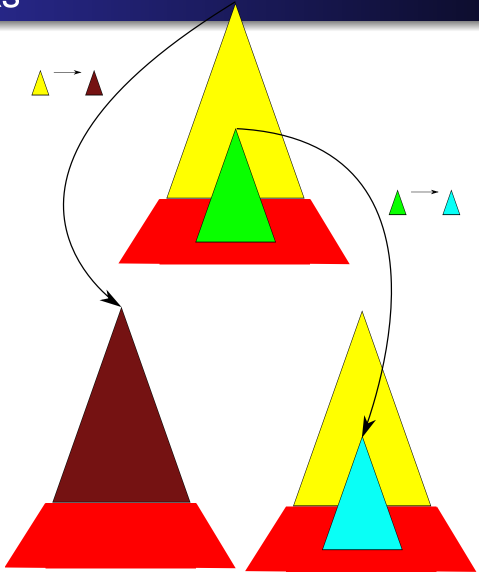
Conversion: $u \overset{*}{\longleftrightarrow} v$

Local peak: $u \longleftarrow s \longrightarrow v$

Joinability: $u \overset{*}{\longrightarrow} t \overset{*}{\longleftarrow} v$

Church-Rosser:
convertible pairs are joinable.

Newmann:
Assume plain rewriting terminates. Then it is Church-Rosser iff every local peak is joinable.



Knuth-Bendix: joinability of critical peaks is just enough for terminating plain rewriting

Normal Rewriting Systems (R, S, E)

Definition: $s \xrightarrow[R\downarrow_{S_E}]{p} t$ iff $s = s\downarrow_{S_E} \xrightarrow[R_{SE}]{p} u \xrightarrow[S_E]{!} u\downarrow_{S_E} = t$

General Assumptions

- (a) S is a Church-Rosser set of rules mod E
- (b) $R_{SE} \cup S_E$ is terminating,
- (c) Rules in R are S_E-normalized,

For Nipkow's higher-order rewriting:

E is α-conversion
S is made of β-reduction and η-expansion
R is made of rules $l \rightarrow r$ such that l and r have the same base type and l is a pattern [Miller].

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Example : commutative groups

$$R = \{ \quad x + x^{-1} \rightarrow 0 \quad \}$$

$$S = \{ \quad x + 0 \rightarrow x \quad \}$$

$$E = \{ \begin{array}{rcl} (x+y)+z & = & x+(y+z) \\ x+y & = & y+x \end{array} \}$$

Example : differentiation at higher types

$$R = \{ \begin{array}{rcl} diff(sin \circ f) & \rightarrow & -cos * diff(f) \\ diff(cos \circ f) & \rightarrow & sin * diff(f) \\ diff(\lambda x.x) & \rightarrow & \lambda x.1 \end{array} \}$$

$$S = \{ \begin{array}{rcl} \lambda x.@(u, x) & \rightarrow & u \\ & & \text{if } x \notin \mathcal{V}ar(u) \\ @(\lambda x.u, v) & \rightarrow & u\{x \mapsto v\} \end{array} \}$$

$$E = \{ \begin{array}{rcl} \lambda y.u\{x \mapsto y\} & = & \lambda x.u \\ & & \text{if } y \notin \mathcal{V}ar(\lambda x.u) \end{array} \}$$

Main property expected from normal rewriting

Conversion: $t_1 \overset{*}{\underset{R \cup S \cup E}{\longleftrightarrow}} t_2$

Joinability: $t_1 \overset{!}{\underset{S_E}{\longrightarrow}} \; \overset{*}{\underset{R \downarrow_{S_E}}{\longrightarrow}} u \overset{*}{\underset{E}{\longleftrightarrow}} v \overset{*}{\underset{R \downarrow_{S_E}}{\longleftarrow}} \; \overset{!}{\underset{S_E}{\longleftarrow}} t_2$

Church-Rosser: every conversion is joinable.

Theorem (Target result for NRSes)

Let (R, S, E) satisfy (a,b,c), and critical local peaks be joinable. Then normal rewriting is CR.

Further requirements:

- First and higher-order rewriting as instances;
- A proof independent from any term structure.

Abstract Positional Rewriting with R

an abstract set of terms $\mathcal{T}$
a monoid of positions $\mathcal{P}$ equiped with
concatenation $\cdot$, neutral Λ, prefix order $>_{\mathcal{P}}$

A domain P is any downward closed subset of $\mathcal{P}$

Rewrite relations become ternary: $u \xrightarrow{p,P,Q} v$

Successor below p of s: $s \xrightarrow{\geq_{\mathcal{P}} p} t$

In normal form below p: $s = s{\downarrow}^{p}$

Normal form below p of s: $s \xrightarrow{(\geq_{\mathcal{P}} p)^*} s{\downarrow}^{p}$

Rewriting with R modulo E at p

$$\xrightarrow[R_E]{p} \;:=\; \xleftrightarrow[E]{(\geq_{\mathcal{P}} p)^*} \xrightarrow[R]{p}$$

Disjoint redexes axiom

$$\xleftarrow[R'_{E'}]{p} \xrightarrow[R_E]{q} \;\subseteq\; \xrightarrow[R_E]{p} \xleftarrow[R'_{E'}]{q} \quad \text{if} \quad p \# q$$

Ancestor redex axiom

$$\xleftarrow[R']{p,P} \xrightarrow[R_E]{p \cdot q} \;\subseteq\; \xrightarrow[R_E]{(>_{\mathcal{P}} p \cdot Q)^*} \xleftarrow[R']{p} \xleftarrow[R_E]{(>_{\mathcal{P}} p \cdot P)^*} \quad \text{if} \quad q >_{\mathcal{P}} P$$

Modulo on left is not allowed !

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Modulo on left is not allowed !

Critical pairs modulo

$$u \xleftarrow[R']{p,P} s \xrightarrow[R_E]{p \cdot q} v \quad \text{with } q \in P$$

Again, position q should not be lost in u, which might happen if R' were a modulo step.

E-steps below p can be allowed provided they do not occur strictly above q.

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Abstract Positional Fringe Rewriting with (R, E)

A *fringe* of $s \xrightarrow[R]{p,P} t$ is a set Q of dis. pos. of P s.t.

$$v \xleftarrow[R]{p,P} \xleftrightarrow[E]{(\geq_{\mathcal{P}}\, p \cdot Q)^*} u \xrightarrow[R_E]{p \cdot q} w \text{ with } q \in P \text{ implies } q \not\geq_{\mathcal{P}} Q.$$

Maximal positions in P form a non-trivial fringe.

We use P^f for an arbitrary fringe

Abstract Positional Fringe Rewriting:

$$\xrightarrow[R_E^f]{p,P} := \xleftrightarrow[E]{(\geq_{\mathcal{P}} p \cdot P^f)^*} \xrightarrow[R]{p,P}$$

Fringe rewriting satisfies a variant of Ara:

$$\xleftarrow[R_E^f]{p,P} \xrightarrow[R_E]{q >_{\mathcal{P}} P_p} \subseteq \xrightarrow[R_E]{(\geq p)^*} \xleftarrow[R_E]{(\geq p)^*}$$

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Normal APR with NARS (R, S, E), E symmetric

(i) *Simplification* is *Church-Rosser* below any p:

$$s \xleftrightarrow[SE]{(\geq_{\mathcal{P}} p)^*} t \text{ iff } s \xrightarrow[S_E]{(\geq_{\mathcal{P}} p)^*} \xleftrightarrow[E]{(\geq_{\mathcal{P}} p)^*} \xleftarrow[S_E]{(\geq_{\mathcal{P}} p)^*} t$$

(ii) $\succ := (\longrightarrow_{R_S} \cup \longrightarrow_S)$ is *E-terminating*

(iii) *Normal rewriting at $p \geq_{\mathcal{P}} q$* is defined as:

$$s \xrightarrow[R_{S_E\downarrow}]{(p,q)} t \;:=\; s = s{\downarrow}^q_{S_E} \xrightarrow[R_{SE}]{p} u \xrightarrow[S_E]{!} u{\downarrow}^q_{S_E} = t$$

normal rewriting at p : take $q = \wedge$

Critical patterns for normal rewriting

Rewrite peak

$$v \xleftarrow[R^f_{SE}]{p,P} u \xrightarrow[R_{SE}]{p\cdot q} w \quad \text{s.t.} \quad q \in P \text{ and } u = u{\downarrow}^p_{S_E}$$

Equational cliff

$$v \xleftrightarrow[E]{p,P} u \xrightarrow[R_{SE}]{p\cdot q} w \quad \text{s.t.} \quad q \in P \setminus \{\wedge\}$$

Simplification cliff

$$v \xleftarrow[S]{p,P} u \xrightarrow[R_{SE}]{p\cdot q} w \ \text{ s.t. } \ q \in P \setminus \{\wedge\} \text{ and } u = u{\downarrow}^q_{S_E}$$

Simplification peak

$$v \xleftarrow[R^f_{SE}]{p,P} u \xrightarrow[S_E]{p\cdot q} w \quad \text{s.t.} \quad q \in P \setminus \{\geq_{\mathcal{P}} P^f\}$$

Church-Rosser theorem for NARSes

Definition

E-joinability:

$$v{\downarrow}_{S_E} \xrightarrow[R_{SE}\cup S_E]{*} s \xleftrightarrow[E]{*} t \xleftarrow[S_E\cup R_{SE}]{*} w{\downarrow}_{S_E}$$

Fringe-*E*-joinability at *p*:

$$v{\downarrow}_{S_E} \xrightarrow[R_{SE}\cup S_E]{*} s \xleftrightarrow[E]{*} t \xleftarrow[S_E\cup R_{SE}]{*} \xleftarrow[R^f_{SE}]{p} w{\downarrow}_{S_E}$$

Theorem (CR NARSes)

A NARS (R, S, E) satisfying (a,b,c) whose critical simplification peaks are fringe-E-joinable is CR iff its critical rewrite peaks, equational and simplification cliffs are E-joinable.

Church-Rosser theorem for NARSes

Definition

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Fringe-*E*-joinability at *p*:

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Theorem (CR NARSes)

A NARS (R, S, E) satisfying (a,b,c) whose critical simplification peaks are fringe-E-joinable is CR iff its critical rewrite peaks, equational and simplification cliffs are E-joinable.

Proof

By rewriting local peaks in conversions, interpreted by a multiset of binary words over the alphabet of terms, and compared in the ordering $((\succ_E)_{lex})_{mul}$.

Elementary steps contribute to proofs with one or two words:

$u \longrightarrow_{R_{SE}} v$ with uv

$u \longrightarrow_{S_E} v$ with vu

$u \longleftrightarrow_E v$ with uv and vu

New: the measure on proofs does not use $(\succ \cup \rhd)_E$

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Plain first-order rewriting with R

Definition

Given $l \rightarrow r, g \rightarrow d \in R$ and $p \in \mathcal{FP}os(l)$ s.t. $l|_p = g$ has mgu σ,
$r\sigma \xleftarrow{\Lambda} l\sigma = (l[g]_p)\sigma \xrightarrow{p} (l[d])\sigma$ is a critical peak of $g \rightarrow d$ onto $l \rightarrow d$ at position p.

Theorem (Knuth and Bendix, 1969)

A terminating rewrite system R is Church-Rosser iff its critical peaks are joinable.

First-order rewriting modulo with (R, E)

Definition

Given $l \to r, g \to d \in R$, $p \in \mathcal{FP}os(l)$, σ a most general E-unifier of the equation $l|_p = g$, then $r\sigma \xleftarrow{\Lambda} l\sigma \overset{(\geq_{\mathcal{P}} p)^*}{\underset{E}{\longleftrightarrow}} (l[g]_p)\sigma \xrightarrow{p} (l[d])\sigma$, is an E-critical peak of $g \to d$ onto $l \to d$ at p.

Definition

Given an equation $l = r \in E$, a rule $g \to d \in R$ and a position $p \in \mathcal{FP}os(l) \setminus \{\wedge\}$ s.t. $l|_p = g$ is unifiable, $l[g] \to l[d]$ is an E-extension of $g \to d$ onto $l = r$ at p.

Theorem (Jouannaud and Kirchner, 1986)

Assume R is E-terminating and closed under E-extensions. Then R is CR modulo E iff its E-critical peaks are E-joinable.

New: no need for finite E-congruence classes !

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Normal first-order rewriting (R, S, E)

Definition

Given $l \to r \in R$, $g \to d \in S$ and $p \in \mathcal{FP}os(g) \setminus \{\wedge\}$ s.t. l and $g|_p$ are SE-unifiable, then $g[l]_p\downarrow \to g[r]_p\downarrow$ is a *S-extension* of $l \to r$ at p.

Definition

Given rules $l \to r \in R$ and $g \to d \in S$, and a position $p \in \mathcal{FP}os(l)$ s.t. σ is a most general E-unifier of $l|_p = g$, then $\{(l[d]_q)\sigma\downarrow \to (r\sigma)\downarrow$ is a simplification pair of $g \to d$ onto $l \to r$ at q.

Theorem

Assume that $R_{SE} \cup S_E$ is E-terminating, S is CR modulo E, and (R, S, E) is closed under (normalized) E-extensions, S-extensions and simplification pairs. Then, normal rewriting is CR iff its SE-critical pairs are E-joinable.

Here, we need finite complete sets of most general unifiers for both E and SE. For an example, E is AC and S is ZI.
Application: Commutative group theory, Polynomials over a commutative ring.

Nipkow's Higher-order rewriting at simple types

- E is α-conversion
- $S = \{\beta, \eta^{-1}\}$
- R is a set of base type higher-order rules in β-long normal form which lhs are patterns
- E-unification: plain unification up to variable renaming of bound variables
- SE-unification: higher-order unification
- Termination of $R_{\beta\eta^{-1}} \cup \{\beta\eta^{-1}\}$ modulo α-conv see [Jouannaud, Rubio, TOCL to appear]
- S is CR modulo α-conversion

Nipkow's higher-order rewriting at simple types

- E-extensions: none
- S-extensions: none since rules are at a base type and only strict subterm of β is an abstraction
- Simplification peaks: none, since lefthand sides are normalized and subterms $@(X, \overline{x})$ are on the fringe in pattern instances.

Theorem

Assume $R_{\beta\eta} \cup \beta\eta^{-1}$ terminates. Then higher-order rewriting is Church-Rosser iff its higher-order critical pairs are joinable.

The difference is that η is now oriented as a reduction, its lefthand side being $\lambda x@(u, x)$ with $x \notin \mathcal{V}ar(u)$.

But the subterm $@(u, x)$ contains the bound variable x, hence cannot unify with a lefthand side of rule.

We therefore get the same result as before.

We may have (finitely many) β-extensions for each rule in R, each extension decreasing the type of the rule.

Let $o : *, a : o, b : o$ and
$R := \{\lambda x : o.a \to \lambda x : o.b\}$.

Then, the β-extension is $a \to b$.

Higher-unification of patterns in presence of associativity and commutativity has complete sets of general unifiers [Boudet, Contejean].

The general result applies to this case as well.

A clean, flexible framework
for all forms of rewriting
obtained
via novel notions of
abstract positional rewriting
and
fringe rewriting

THANKS