

Normalisation by Completeness with Heyting Algebras

Gaëtan Gilbert^{1,2} and Olivier Hermant³

¹ ENS Lyon, France

`gaetan.gilbert@ens-lyon.fr`

² Inria Paris, France

³ MINES ParisTech, PSL Research University, France

`olivier.hermant@mines-paristech.fr`

Abstract. Usual normalization by evaluation techniques have a strong relationship with completeness with respect to Kripke structures. But Kripke structures is not the only semantics that fits intuitionistic logic: Heyting algebras are a more algebraic alternative.

In this paper, we focus on this less investigated area: how completeness with respect to Heyting algebras generate a normalization algorithm for a natural deduction calculus, in the propositional fragment. Our main contributions is that we prove in a direct way completeness of natural deduction with respect to Heyting algebras, that the underlying algorithm natively deals with disjunction, that we formalized those proofs in Coq, and give an extracted algorithm.

1 Introduction

In logic, a restriction to cut-free proofs makes analysis of a theory and proof-search significantly simpler. Evaluating programs boils down to finding efficient ways to reach a normal form, in order to produce a result.

Through the proof-as-programs paradigms, those two processes can be reduced to a single one: reduction steps of lambda-terms, such as β reduction, can be seen as a way to remove cuts from proofs expressed in natural deduction. Under this correspondence, a proof is cut-free when the associated proof-term is in normal form.

But there exist other, semantic, ways to eliminate cuts from proofs [14], through a reinforcement of the completeness theorem, hereafter strong completeness, in combination with soundness. Sometimes those proofs can be made constructive [16,10,3]. A natural question, therefore, arises: what is the computational content of such proofs ?

A link has already been exhibited. A fruitful line of research, dubbed normalisation by evaluation, aims at evaluating a program in a type-directed fashion, by reusing the reduction mechanisms a hand at the meta

level⁴ through a pair of reflection/reification functions [2]. Soon after, Coquand noticed a strong similarity with completeness proofs [4].

This seminal work has been extended to more complex types [5,1], and also studied from the point of view of the completeness theorem for intuitionistic natural deduction with respect to Kripke-like structures [9,6,8]. But when it comes to incorporate disjunction, one must be very careful, in particular because Kripke structures require worlds to *decide* between both members of the conjunction - from a pure normalization by evaluation point of view, dealing with sum types also require special care.

In this paper, we go on this line, relating constructive completeness proofs and normalization procedures. But, instead of considering Kripke semantics, as has been done in the works described above, we consider *Heyting algebras*:

- *strong* completeness theorems, and therefore cut elimination [7] also hold, and they are constructive from their inception;
- handling disjunction is straightforward, and hence we get cut elimination for sum types.

An adaptation of existing completeness proofs with respect to Heyting algebras is required, since all the known proofs, starting from Okada's contribution to linear logic [11], use sequent calculus.

To support these claims, we have formalized the proofs of this paper in Coq, and used extraction to get an executable interpreter. To keep the complexity of the formalization reasonable, we remained in the propositional fragment. The Coq sources are available at <https://github.com/SkySkimmer/NormalisationByCompleteness>.

Therefore, the organization of this paper is the following: in Sec. 2 we recall natural deduction, insisting on the notion of cut, and show basic lemmas. In Sec. 3, we develop the strong completeness proof, and discuss its Coq formalization in the next Sec. 4.1, where we also devise the behavior of the extracted algorithm on examples. Sec. 5 concludes the paper.

2 Natural Deduction

In this section, we recall the basic definitions and lemmas of natural deduction.

⁴ namely, the programming language in which the evaluation function is written

Definition 1 (Terms and formulas). Let \mathcal{V} be an infinite set of variables and \mathcal{S} be a set of function symbols along with an arity. The set of terms \mathcal{T} is defined by

$$t ::= x \mid f(t_1, \dots, t_n)$$

for $x \in \mathcal{V}$ and $f \in \mathcal{S}$ with arity n .

Let \mathcal{P} be a set of predicate symbols. Formulas \mathcal{F} are defined by

$$A ::= P(t_1, \dots, t_n) \mid A \wedge B \mid A \vee B \mid A \Rightarrow B \mid \top \mid \perp \mid \forall x.A \mid \exists x.A$$

where $P \in \mathcal{P}$ with arity n .

Definition 2 (Substitutions). A substitution σ is a partial function from variables to terms.

We expand it inductively to a function from terms to terms and formulas to formulas, letting $\sigma(x) = x$ for $x \notin \text{dom}(\sigma)$.

Notably for $\mathcal{Q} \in \{\forall, \exists\}$, $\sigma(\mathcal{Q} x.A) := \mathcal{Q}x.\sigma(A)$, assuming x fresh w.r.t. the image of σ by α -conversion. This is always possible since $\text{dom}(\sigma)$ is finite, and so the image of σ is also finite.

Definition 3 (Updated Substitution). Let σ be a substitution, $x \in \mathcal{V}$ and $t \in \mathcal{T}$, $\sigma[x \mapsto t]$ is the substitution with domain $\text{dom}(\sigma) \cup \{x\}$ such that for all $y \neq x$, $\sigma[x \mapsto t](y) = \sigma(y)$ and $\sigma[x \mapsto t](x) = t$.

The substitution with the empty set as domain is denoted \emptyset . For t a term (resp. A a formula), x a variable and u a term, we abbreviate $\emptyset[x \mapsto u](t)$ (resp. $\emptyset[x \mapsto u](A)$) as $t[u/x]$ (resp. $A[t/x]$).

Definition 4 (Contexts). A context Γ is a list of formulas: $\Gamma = [A_1, \dots, A_n]$.

We let Γ, A be the concatenation of Γ and A .

We say that B is in Γ (denoted $B \in \Gamma$) iff there is some i such that B is A_i . A context Γ is included in a context Σ (denoted $\Gamma \subseteq \Sigma$), when any $B \in \Gamma$ is also in Σ .

Remark 1. The relation \subseteq is a preorder, but not an order. Indeed, it strictly subsumes contraction: $\Gamma, A, A \subseteq \Gamma, A$.

Definition 5 (Cut-Free Proofs). Figure 1 defines the relations \vdash_{ne} (neutral proof) and \vdash^* (cut-free proof) by mutual induction.

In Fig. 1, Rules on the left are introduction rules and produce cut-free proofs, while rules on the right are elimination rules and produce neutral proofs. The usual natural deduction calculus NJ is a merge of both relations.

$$\begin{array}{c}
\frac{\Gamma \vdash_{ne} A}{\Gamma \vdash^* A} \text{coerce} \qquad \frac{A \in \Gamma}{\Gamma \vdash_{ne} A} \text{ax} \\
\frac{\Gamma \vdash^* A \quad \Gamma \vdash^* B}{\Gamma \vdash^* A \wedge B} \wedge_I \qquad \frac{\Gamma \vdash_{ne} A \wedge B}{\Gamma \vdash_{ne} A} \wedge_{E_l} \qquad \frac{\Gamma \vdash_{ne} A \wedge B}{\Gamma \vdash_{ne} B} \wedge_{E_r} \\
\frac{\Gamma \vdash^* A}{\Gamma \vdash^* A \vee B} \vee_{I_l} \quad \frac{\Gamma \vdash^* B}{\Gamma \vdash^* A \vee B} \vee_{I_r} \quad \frac{\Gamma \vdash_{ne} A \vee B \quad A, \Gamma \vdash^* C \quad B, \Gamma \vdash^* C}{\Gamma \vdash_{ne} C} \vee_E \\
\frac{\Gamma, A \vdash^* B}{\Gamma \vdash^* A \Rightarrow B} \Rightarrow_I \qquad \frac{\Gamma \vdash_{ne} A \Rightarrow B \quad \Gamma \vdash^* A}{\Gamma \vdash_{ne} B} \Rightarrow_E \\
\frac{}{\Gamma \vdash^* \top} \top_I \qquad \frac{\Gamma \vdash_{ne} \perp}{\Gamma \vdash_{ne} A} \perp_E \\
\frac{\Gamma \vdash^* A \quad x \notin FV(\Gamma)}{\Gamma \vdash^* \forall x.A} \forall_I \qquad \frac{\Gamma \vdash_{ne} \forall x.A}{\Gamma \vdash_{ne} A[t/x]} \forall_E \\
\frac{\Gamma \vdash^* A[t/x]}{\Gamma \vdash^* \exists x.A} \exists_I \qquad \frac{\Gamma \vdash_{ne} \exists x.A \quad A, \Gamma \vdash^* C \quad x \notin FV(C, \Gamma)}{\Gamma \vdash_{ne} C} \exists_E
\end{array}$$

Fig. 1. Rules of Natural Deduction

Definition 6 (Natural Deduction NJ). *The judgment $\Gamma \vdash A$ has the same rules as both $\Gamma \vdash^* A$ and $\Gamma \vdash_{ne} A$. Then if $\Gamma \vdash^* A$ then $\Gamma \vdash A$ and if $\Gamma \vdash_{ne} A$ then $\Gamma \vdash A$.*

Lemma 1 (Weakening). *Let Γ, Σ be two such that $\Gamma \subseteq \Sigma$. Let A be a formula. If $\Gamma \vdash^* A$ (resp. $\Gamma \vdash_{ne} A$, resp. $\Gamma \vdash A$) is derivable, then $\Sigma \vdash^* A$ (resp. $\Sigma \vdash_{ne} A$, resp. $\Sigma \vdash A$) is derivable.*

Proof. By mutual induction on $\Gamma \vdash^* A$ and $\Gamma \vdash_{ne} A$, and by induction on $\Gamma \vdash A$. \square

Corollary 1 (Contraction). *For any context Γ and formula B , if $\Gamma, A, A \vdash B$ then $\Gamma, A \vdash B$.*

Neutral proofs are such that they can replace axioms in cut-free proofs without introducing any cut.

Lemma 2 (Axiom Replacement). *Let Γ, Σ be contexts such that, for each formula $C \in \Gamma$, we can derive $\Sigma \vdash_{ne} C$. Then for any formula A , if $\Gamma \vdash^* A$ (resp. $\Gamma \vdash_{ne} A$) we can derive $\Sigma \vdash^* A$ (resp. $\Sigma \vdash_{ne} A$). Similarly, if for each $C \in \Gamma$ we can derive $\Sigma \vdash C$ then for any formula A such that $\Gamma \vdash A$ we can derive $\Sigma \vdash A$.*

Proof. By mutual induction on $\Gamma \vdash^* A$ and $\Gamma \vdash_{ne} A$, and by induction on $\Gamma \vdash A$. Note that we need the weakening lemma (Lem. 1) when the context is modified in a premise of a rule.

Consider for instance the \Rightarrow_I case of Fig. 1, we know that $\Gamma, A \vdash^* B$ is derivable. For each $C \in \Gamma, A, \Sigma, A \vdash_{ne} C$, using the weakening lemma when $C \in \Gamma$. Then by induction hypothesis, $\Sigma, A \vdash^* B$ and by \Rightarrow_I we conclude $\Sigma \vdash^* A \Rightarrow B$. \square

Lemma 3 (Kleene's Inversion Lemma). *Let Γ be a context, A and B be formulas.*

If $\Gamma \vdash_{ne} A \Rightarrow B$ (resp. $\Gamma \vdash^ A \Rightarrow B$) then $\Gamma, A \vdash_{ne} B$ (resp. $\Gamma, A \vdash^* B$).*

Proof. If $\Gamma \vdash_{ne} A \Rightarrow B$, then by weakening $\Gamma, A \vdash_{ne} A \Rightarrow B$. By *ax* and *coerce* we have $\Gamma, A \vdash^* A$. Then by \Rightarrow_E , $\Gamma, A \vdash_{ne} B$.

If $\Gamma \vdash^* A \Rightarrow B$, we analyze the last rule of the derivation:

- it is *coerce*: the premiss is $\Gamma \vdash_{ne} A \Rightarrow B$, then $\Gamma, A \vdash_{ne} B$ and by *coerce*, $\Gamma, A \vdash^* B$.
- otherwise it is \Rightarrow_I : the premiss is $\Gamma, A \vdash^* B$. \square

3 Strong completeness by Heyting Algebras

3.1 Heyting Algebras

Definition 7 (Complete Lattice). *A complete lattice is a tuple*

$$\mathcal{A} = (A, \leq, \wedge, \vee, \top, \perp, \bigwedge, \bigvee)$$

such that (A, \leq) is a partial order with binary meet \wedge and join \vee , arbitrary meet \bigwedge and join \bigvee and global maximum \top and minimum \perp .

Definition 8 (Complete Heyting Algebra). *A Heyting algebra is a structure $\mathcal{H} = (H, \leq, \wedge, \vee, \Rightarrow, \top, \perp, \bigwedge, \bigvee)$ such that $(H, \leq, \wedge, \vee, \top, \perp, \bigwedge, \bigvee)$ is a complete lattice and verifies the implication property*

$$\forall a, b, c, a \leq b \Rightarrow c \text{ if and only if } a \wedge b \leq c$$

Lemma 4. *In a Heyting algebra, binary meet and join distribute over each other.*

Proof. Let $a, b, c \in H$

- $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$:
 $a \wedge b \leq (a \wedge b) \vee (a \wedge c)$ and $a \wedge c \leq (a \wedge b) \vee (a \wedge c)$.
 By the implication property,

$$b \leq a \Rightarrow ((a \wedge b) \vee (a \wedge c))$$

and

$$c \leq a \Rightarrow ((a \wedge b) \vee (a \wedge c))$$

Then $b \vee c \leq a \Rightarrow ((a \wedge b) \vee (a \wedge c))$ and we conclude by the implication property.

- $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$: true in all lattices
- $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$: true in all lattices
- $(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$:

By the implication property, equivalent to $a \vee b \leq (a \vee c) \Rightarrow (a \vee (b \wedge c))$

$$\iff a \leq (a \vee c) \Rightarrow (a \vee (b \wedge c)) \text{ and } b \leq (a \vee c) \Rightarrow (a \vee (b \wedge c))$$

$$\iff a \wedge (a \vee c) \leq a \vee (b \wedge c) \text{ (true) and } b \wedge (a \vee c) \leq a \vee (b \wedge c)$$

$$\iff a \vee c \leq b \Rightarrow (a \vee (b \wedge c))$$

$$\iff a \leq b \Rightarrow (a \vee (b \wedge c)) \text{ and } c \leq b \Rightarrow (a \vee (b \wedge c))$$

$$\iff a \wedge b \leq a \vee (b \wedge c) \text{ (true) and } c \wedge b \leq a \vee (b \wedge c) \text{ (true)} \quad \square$$

Definition 9 (Interpretation). A valuation on a set \mathcal{D} , called the domain, is a partial function $\varphi : \mathcal{V} \rightarrow \mathcal{D}$ with finite support. The syntax for updating valuations is the same as in Def. 3.

A first-order model on a Heyting algebra \mathcal{H} is given by a domain \mathcal{D} , and for each $f \in \mathcal{S}$ function symbol of arity n a function $\llbracket f \rrbracket : \mathcal{D}^n \rightarrow \mathcal{D}$ and for each predicate symbol $P \in \mathcal{P}$ of arity n a function $\llbracket P \rrbracket : \mathcal{D}^n \rightarrow \mathcal{H}$.

Let φ be a valuation, t a term and A a formula. Assume that $FV(t) \cup FV(A) \subseteq \text{dom}(\varphi)$. The interpretations $\llbracket t \rrbracket_\varphi \in \mathcal{D}$ and $\llbracket A \rrbracket_\varphi \in \mathcal{H}$ are defined in the usual inductive way.

Notably:

$$\llbracket P(t_1, \dots, t_k) \rrbracket_\varphi := \llbracket P \rrbracket(\llbracket t_1 \rrbracket_\varphi, \dots, \llbracket t_k \rrbracket_\varphi)$$

$$\llbracket \forall x. A \rrbracket_\varphi := \bigwedge_{v \in \mathcal{D}} \{ \llbracket A \rrbracket_{\varphi[x \mapsto v]} \}$$

$$\llbracket \exists x. A \rrbracket_\varphi := \bigvee_{v \in \mathcal{D}} \{ \llbracket A \rrbracket_{\varphi[x \mapsto v]} \}$$

Theorem 1 (Soundness). Let Γ be a context and A be a formula, if $\Gamma \vdash A$ is derivable in NJ then for any Heyting algebra \mathcal{H} , for any model on \mathcal{H} and valuation φ ,

$$\llbracket \Gamma \rrbracket_\varphi \leq \llbracket A \rrbracket_\varphi$$

where $\llbracket \Gamma \rrbracket_\varphi := \bigwedge_{C \in \Gamma} \llbracket C \rrbracket_\varphi$.

Proof. Standard induction [13]. □

3.2 Completeness

We now proceed to the construction of a universal Heyting algebra, that is suitable for *cut-free*, or *strong* (in the sense of [12]) completeness. This is in contrast with more usual Lindenbaum constructions [13], where the algebra is formed with (provability-) equivalence classes of formulas.

Definition 10 (Extraction). *The extraction of a formula A is*

$$\lfloor A \rfloor := \{\Gamma, \Gamma \vdash^* A\}$$

Definition 11 (Universal Heyting Algebra). *The underlying set of the universal Heyting algebra (aka the context algebra) is:*

$$\Omega := \left\{ \bigcap_{i \in I} \lfloor A_i \rfloor, (A_i)_{i \in I} \text{ family of formulas} \right\}$$

That is to say, the closure by arbitrary intersections of formula extractions. Ω is partially ordered by inclusion. The operations are defined as:

$$\begin{aligned} a \leq b &:= a \subseteq b \\ a \wedge b &:= a \cap b \\ \bigwedge A &:= \bigcap A \\ a \vee b &:= \bigcap \{\omega \in \Omega, a \cup b \leq \omega\} \\ \bigvee A &:= \bigcap \{\omega \in \Omega, \bigcup A \leq \omega\} \\ a \Rightarrow b &:= \bigvee \{c \in \Omega, a \wedge c \leq b\} \\ \top &:= \{\Gamma, \Gamma \text{ context}\} = \lfloor \top \rfloor \\ \perp &:= \{\Gamma, \forall A, \Gamma \vdash A\} = \lfloor \perp \rfloor \end{aligned}$$

By abuse of notation, this algebra is noted Ω as well.

\wedge and \bigwedge are clearly greatest lower bounds in Ω wrt the order \leq , as well as \vee and \bigvee are lowest upper bounds. As well, we can simplify a little bit lowest upper bounds, thanks to the following lemma:

Lemma 5. *The following identities are verified:*

$$\begin{aligned} a \vee b &= \bigcap \{\lfloor D \rfloor, a \cup b \leq \lfloor D \rfloor, D \text{ formula}\} \\ \bigvee A &= \bigcap \{\lfloor D \rfloor, \bigcup A \leq \lfloor D \rfloor, D \text{ formula}\} \\ a \Rightarrow b &= \bigcap \{\lfloor D \rfloor, \bigcup \{c \in \Omega, a \wedge c \leq b\} \leq \lfloor D \rfloor, D \text{ formula}\} \end{aligned}$$

Proof. :

- as \vee is the lowest upper bound, $a \vee b \leq \bigcap \{\lfloor D \rfloor, a \cup b \leq \lfloor D \rfloor, D \text{ formula}\}$.
Conversely, let ω such that $a \cup b \leq \omega$. Since $\omega \in \Omega$, $\omega = \bigcap_{i \in I} \lfloor C_i \rfloor$ for some $(C_i)_{i \in I}$. For all $i \in I$, $a \cup b \leq \lfloor C_i \rfloor$, and therefore $\omega \leq \bigcap \{\lfloor D \rfloor, a \cup b \leq \lfloor D \rfloor\}$.
- Same as above.
- Similarly, since $a \Rightarrow b$ is a lowest upper bound. □

Lemma 6. *Let $\omega \in \Omega$, and $\Gamma \in \omega$. Then, for any context Δ , $\Delta, \Gamma \in \omega$.*

Proof. By applying Lem. 1 to Def. 10 and Def. 11. \square

Lemma 7. Ω forms a Heyting algebra.

Proof. Ω is closed by arbitrary intersection and $\forall A, \lfloor A \rfloor \in \Omega$, so the operations all produce values in Ω . As already said, \leq is an order for which \wedge and \bigwedge are greatest lower bounds, and \vee and \bigvee are lowest upper bounds. \top and \perp are trivially the greatest and least element, respectively. It remains to check the implication property:

- Assume $a \leq b \Rightarrow c$, with $c = \bigcap_{k \in K} \lfloor C_k \rfloor$. Let $\Gamma \in a \wedge b$ and $k \in K$, we want to show $\Gamma \in \lfloor C_k \rfloor$, that is to say $\Gamma \vdash^* C_k$.
 $\Gamma \in a$ so $\Gamma \in b \Rightarrow c$ and we have for any D , if $\bigcup \{e \in \Omega, b \wedge e \leq c\} \leq \lfloor D \rfloor$ then $\Gamma \in \lfloor D \rfloor$.
 Let us show that $D := \Gamma \Rightarrow C_k$ verifies this hypothesis, where $\Gamma \Rightarrow B := A_1 \Rightarrow \dots \Rightarrow A_n \Rightarrow B$ (with $\Gamma = A_1, \dots, A_n$ and B formula).
 Let $e \in \Omega$ with $b \wedge e \leq c$. Let $\Delta \in e$, $\{\Delta, \Gamma\} \in b \wedge e$ by Lem. 1, then $\Delta, \Gamma \in c$, and $\Delta, \Gamma \vdash^* C_k$.
 By \Rightarrow_I , $\Delta \vdash^* \Gamma \Rightarrow C_k$, that is to say $\Delta \vdash^* D$. This holds for any such Δ , so $e \leq \lfloor D \rfloor$, and D verifies the desired hypothesis.
 Therefore $\Gamma \vdash^* \Gamma \Rightarrow C_k$ and by repeated application of Lem. 3 and Lem. 1 $\Gamma \vdash^* C_k$.
 Finally, $\Gamma \in c$.
- Conversely, assume $a \wedge b \leq c$, then $a \leq \bigvee \{e, e \wedge b \leq c\} = b \Rightarrow c$. \square

Definition 12 (Interpretation in the Context Algebra). *The model on Ω is defined as:*

- the domain is the set of terms
- if f is a function symbol of arity n ,

$$\llbracket f \rrbracket := (t_1, \dots, t_n) \mapsto f(t_1 \dots t_n)$$

- if P is a predicate symbol of arity n ,

$$\llbracket P \rrbracket := (t_1, \dots, t_n) \mapsto \lfloor P(t_1, \dots, t_n) \rfloor$$

An easy consequence of this lemma is the following, where we implicitly coerce valuations with their underlying substitution.

Lemma 8. *For any t and valuation φ , $\llbracket t \rrbracket_\varphi = \varphi(t)$.*

Proof. By induction. \square

Definition 13 (Closure). For any A formula, we let the closure of A be

$$cl(A) := \bigcap \{d \in \Omega, [A] \in d\}$$

Lemma 9. For any A , $cl(A) \in \Omega$.

Proof. Ω is stable by arbitrary intersection. \square

Lemma 10.

$$[A] \in cl(A)$$

Proof. $cl(A)$ is the lowest upper bound of all d containing $[A]$. \square

Lemma 11. For any A , $cl(A) = \bigcap \{[D], [A] \in [D]\}$.

Proof. Similar to the proof of Lem. 5. \square

Then $\Gamma \in cl(A)$ means for all formula D , if $[A] \vdash^* D$ then $\Gamma \vdash^* D$. In a sense, the members of $cl(A)$ verify the axiom replacement lemma, except that this new operation does not necessarily preserve the structure of the derivation. $\Gamma \in cl(A)$ is a weaker statement than $\Gamma \vdash_{ne} A$:

Lemma 12. For Γ context and A formula, if $\Gamma \vdash_{ne} A$ then $\Gamma \in cl(A)$.

Proof. By Lem. 2, considering the previous Lem. 11. \square

Theorem 2 (Key theorem). For any A formula and σ valuation into Ω , σ is also a substitution and

$$cl(\sigma(A)) \leq \llbracket A \rrbracket_\sigma \leq \lfloor \sigma(A) \rfloor$$

Proof. For clarity, we omit the valuation/substitution σ when it plays no role. The proof is done by induction on A :

- A is atomic: $\llbracket A \rrbracket = \lfloor A \rfloor$, so we only need to check $cl(A) \leq \lfloor A \rfloor$. Let $\Gamma \in cl(A)$. as we have $A \vdash^* A$, by definition of $cl(A)$, we have $\Gamma \vdash^* A$ and therefore $\Gamma \in \lfloor A \rfloor$.
- $cl(A \wedge B) \leq \llbracket A \wedge B \rrbracket$: by induction hypothesis we only need to show $cl(A \wedge B) \leq cl(A) \cap cl(B)$.
Let $\Gamma \in cl(A \wedge B)$ and D such that $A \vdash^* D$ (resp. $B \vdash^* D$). Since $A \wedge B \vdash_{ne} A$ (resp. $A \wedge B \vdash_{ne} B$), by Lem. 2 we have $\Gamma \vdash^* D$ and $\Gamma \in cl(A)$ (resp. $\Gamma \in cl(B)$).

$\llbracket A \wedge B \rrbracket \leq \lfloor A \wedge B \rfloor$: by the induction hypothesis we have $\llbracket A \wedge B \rrbracket \leq \lfloor A \rfloor \cap \lfloor B \rfloor$. The \wedge_I rule concludes the proof.

- $cl(A \vee B) \leq \llbracket A \vee B \rrbracket$: consider C such that $\llbracket A \rrbracket \cup \llbracket B \rrbracket \leq \lfloor C \rfloor$. We have to show $\llbracket A \vee B \rrbracket \in \lfloor C \rfloor$.
Since, by Lem. 10 and induction hypothesis, $\llbracket A \rrbracket \in cl(A) \leq \lfloor C \rfloor$ (resp. $\llbracket B \rrbracket \in cl(B) \leq \lfloor C \rfloor$), we have $A \vdash^* C$ (resp. $B \vdash^* C$). Then by \vee_E and *coerce* we have $A \vee B \vdash^* C$.

$\llbracket A \vee B \rrbracket \leq \lfloor A \vee B \rfloor$: by definition of $\llbracket A \rrbracket \vee \llbracket B \rrbracket$, we need to show that $\llbracket A \rrbracket \cup \llbracket B \rrbracket \leq \lfloor A \vee B \rfloor$.

By induction hypothesis, $\llbracket A \rrbracket \cup \llbracket B \rrbracket \leq \lfloor A \rfloor \cup \lfloor B \rfloor$, then the \vee_I rule concludes.

- $cl(A \Rightarrow B) \leq \llbracket A \Rightarrow B \rrbracket$: by the implication rule we need $cl(A \Rightarrow B) \wedge \llbracket A \rrbracket \leq \llbracket B \rrbracket$, and by induction hypothesis, it is sufficient to show $cl(A \Rightarrow B) \wedge \lfloor A \rfloor \leq cl(B)$.
Let $\Gamma \in cl(A \Rightarrow B) \wedge \lfloor A \rfloor$, that is to say:

$$\Gamma \vdash^* A \text{ and for any } C, \text{ if } A \Rightarrow B \vdash^* C \text{ then } \Gamma \vdash^* C$$

Let D such that $B \vdash^* D$. To show $\Gamma \vdash^* D$, we first show that $\Gamma \vdash^* \Gamma \Rightarrow D$. We have, by hypothesis and Lem. 1, the following proof:

$$\frac{\frac{A \Rightarrow B, \Gamma \vdash_{ne} A \Rightarrow B}{A \Rightarrow B, \Gamma \vdash_{ne} A} \quad A \Rightarrow B, \Gamma \vdash^* A}{A \Rightarrow B, \Gamma \vdash_{ne} B}$$

So, by Lem. 2, $A \Rightarrow B, \Gamma \vdash^* D$, and by repeated \Rightarrow_I , $A \Rightarrow B \vdash^* \Gamma \Rightarrow D$. By hypothesis on Γ , $\Gamma \vdash^* \Gamma \Rightarrow D$. By a repeated application of Lem. 3 and Lem. 1, we get $\Gamma \vdash^* D$.

$\llbracket A \Rightarrow B \rrbracket \leq \lfloor A \Rightarrow B \rfloor$: by induction hypothesis, $\llbracket B \rrbracket \leq \lfloor B \rfloor$, so $\llbracket A \rrbracket \Rightarrow \llbracket B \rrbracket \leq \llbracket A \rrbracket \Rightarrow \lfloor B \rfloor$ by the intersection (with $\llbracket A \rrbracket$) and the implication properties. By induction hypothesis also, $cl(A) \leq \llbracket A \rrbracket$, and therefore $cl(A) \wedge (\llbracket A \rrbracket \Rightarrow \lfloor B \rfloor) \leq \llbracket A \rrbracket \wedge (\llbracket A \rrbracket \Rightarrow \lfloor B \rfloor) \leq \lfloor B \rfloor$, that is to say $\llbracket A \rrbracket \Rightarrow \lfloor B \rfloor \leq cl(A) \Rightarrow \lfloor B \rfloor$.

All in all, $\llbracket A \Rightarrow B \rrbracket \leq cl(A) \Rightarrow \lfloor B \rfloor$, and showing $cl(A) \Rightarrow \lfloor B \rfloor \leq \lfloor A \Rightarrow B \rfloor$ suffices.

Let c such that $cl(A) \wedge c \leq \lfloor B \rfloor$, we show that $\lfloor A \Rightarrow B \rfloor$ is an upper bound for c , so let $\Gamma \in c$. By Lem. 1 $A, \Gamma \in cl(A) \wedge c$, and $A, \Gamma \vdash^* B$, so by \Rightarrow_I , $\Gamma \vdash^* A \Rightarrow B$. This holds for any c , so $cl(A) \Rightarrow \lfloor B \rfloor \leq \lfloor A \Rightarrow B \rfloor$.

- \top and \perp are trivial cases.
- $cl(\sigma(\forall x.A)) \leq \llbracket \forall x.A \rrbracket_\sigma$:
Without loss of generality, we assume $\sigma(\forall x.A) = \forall x.\sigma(A)$ (see Def. 2).
Let $\Gamma \in cl(\sigma(\forall x.A))$.

We need to prove that for all term d , $\Gamma \in \llbracket A \rrbracket_{\sigma[x \mapsto d]}$. Let d a term, showing $\Gamma \in cl(\sigma[x \mapsto d](A))$ suffices by induction hypothesis.

Let D such that $\sigma[x \mapsto d](A) \vdash^* D$. As x does not appear in the image of σ , $\sigma[x \mapsto d](A) = (\sigma(A))[d/x]$, and we have:

$$\frac{\frac{}{[\forall x.\sigma(A)] \vdash_{ne} \forall x.\sigma(A)} \alpha x}{[\forall x.\sigma(A)] \vdash_{ne} \sigma[x \mapsto d](A)} \forall_E$$

Then by Lem. 2, $[\forall x.\sigma(A)] \vdash^* D$. As we assumed $\Gamma \in cl(\forall x.\sigma(A))$, the claim follows.

$\llbracket \forall x.A \rrbracket_{\sigma} \leq \lfloor \sigma(\forall x.A) \rfloor$:

Let $\Gamma \in \llbracket \forall x.A \rrbracket_{\sigma}$, where by α -conversion we assume x fresh.

By Def. 9, $\Gamma \in \llbracket A \rrbracket_{\sigma[x \mapsto x]}$. By induction hypothesis we conclude $\Gamma \in \lfloor \sigma[x \mapsto x](A) \rfloor$.

Finally, by the \forall_I rule $\Gamma \vdash^* \forall x.\sigma[x \mapsto x](A)$, and by freshness of x , $\Gamma \vdash^* \sigma(\forall x.A)$.

– $cl(\sigma(\exists x.A)) \leq \llbracket \exists x.A \rrbracket_{\sigma}$:

Let $\Gamma \in cl(\sigma(\exists x.A))$, assuming x fresh. By Lem. 5, $\Gamma \in \llbracket \exists x.A \rrbracket_{\sigma}$ if and only if for all D , such that for each term d $\llbracket A \rrbracket_{\sigma[x \mapsto d]} \leq \lfloor D \rfloor$, then $\Gamma \vdash^* D$. Let such a D , we give a derivation of $[\exists x.\sigma(A)] \vdash^* D$, which allows to conclude by assumption on Γ .

By induction hypothesis, $[\sigma(A)] \in cl(\sigma(A)) \leq \llbracket A \rrbracket_{\sigma}$, and by hypothesis on D , $[\sigma(A)] \in \lfloor D \rfloor$. With Lem. 1, we get a proof of $\exists x.\sigma(A), \sigma(A) \vdash^* D$. As $\exists x.\sigma(A) \vdash_{ne} \exists x.\sigma(A)$ has a neutral proof, we can build the desired derivation:

$$\frac{\exists x.\sigma(A), \sigma(A) \vdash^* D \quad \exists x.\sigma(A) \vdash_{ne} \exists x.\sigma'(A)}{\exists x.\sigma(A) \vdash_{ne} D} \exists_E$$

$\llbracket \exists x.A \rrbracket_{\sigma} \leq \lfloor \sigma(\exists x.A) \rfloor$, assuming x fresh in the image of σ :

We show that $\lfloor \sigma(\exists x.A) \rfloor$ is an upper bound for all $\llbracket A \rrbracket_{\sigma[x \mapsto d]}$, where d is any term. This allows to conclude.

Let d, Γ , such that $\Gamma \in \llbracket A \rrbracket_{\sigma[x \mapsto d]}$. By induction hypothesis $\Gamma \in \lfloor \sigma[x \mapsto d](A) \rfloor$.

$\sigma[x \mapsto d](A) = (\sigma(A))[d/x]$, so $\Gamma \vdash^* (\sigma(A))[d/x]$ and by the \Rightarrow_I rule, $\Gamma \vdash^* \exists x.\sigma(A)$, i.e. $\Gamma \vdash^* \sigma(\exists x.A)$. \square

Theorem 3 (Strong Completeness). *Let Γ be a context and A a formula, if for any Heyting algebra, any model and any valuation φ , $\llbracket \Gamma \rrbracket_{\varphi} \leq \llbracket A \rrbracket_{\varphi}$, then $\Gamma \vdash^* A$.*

Proof. We apply the hypothesis on the universal algebra of Def. 11, the interpretation of Def. 12 and the empty valuation/substitution.

Consider $C \in \Gamma$. $C \in cl(C)$ by Lem. 10. By Lem. 6 and Thm. 2, $\Gamma \in cl(C) \leq \llbracket C \rrbracket$. So $\Gamma \in \llbracket \Gamma \rrbracket \leq \llbracket A \rrbracket$. By Thm. 2, $\llbracket A \rrbracket \leq \lfloor A \rfloor$. Finally $\Gamma \vdash^* A$.

Theorem 4 (Cut Elimination). *Let Γ be a context and A a formula. If $\Gamma \vdash A$, then $\Gamma \vdash^* A$.*

Proof. By soundness (Thm. 1) and strong completeness (Thm. 3).

4 The algorithm in Practice

This work has been formalized in Coq for the propositional fragment, so as to focus on the core of the algorithm, without dealing with binders.

4.1 Formalization: the Algorithm

If we naively try to define Ω as $\{\{\Gamma : \text{context}, \forall A, \mathcal{P}(A) \rightarrow \Gamma \vdash^* A\} \mid \mathcal{P} : \text{form} \rightarrow \text{Type}\}$, where \mathcal{P} is an index predicate for the formulas A_i , then allowing arbitrary intersections of elements of Ω leads to universe inconsistencies due to impredicativity.

Using *Prop* instead of *Type* is a solution that allows the formalization. But as a consequence, the predicate $\Gamma \vdash^* A$ lives in *Prop*, which means that we cannot extract anything due to proof irrelevance.

Moreover, using the *Eval compute* command on the algorithm applied to a derivation we can observe the behavior of the algorithm. However, since formulas are processed by Thm. 2 which performs case analysis, computation stalls if the derivation involves formula variables.

A workaround, that allowed us to extract a (possibly unsound) algorithm, is to relax the universe constraint, and to remain in *Type*, deliberately working in an inconsistent system.

Three difficulties obfuscate the investigation of the algorithm (see the proofs of Thm. 3 and Thm. 4):

- the $\Gamma \in cl(\Gamma)$ step involves a conjunction of formula closures, and calls technical lemmas. This step can be avoided by considering empty contexts, i.e. $\Gamma = []$ and $\llbracket \Gamma \rrbracket = \top$.
- the $\llbracket A \rrbracket \subseteq \lfloor A \rfloor$ and $cl(\Gamma) \subseteq \llbracket \Gamma \rrbracket$ steps, i.e. calling Thm. 2, the key theorem, that in many cases makes a very indirect use of the NJ rules, potentially appealing to inversion results (Lem. 3).

- the $\llbracket \Gamma \rrbracket \subseteq \llbracket A \rrbracket$ step, i.e. soundness of NJ with respect to Ω . It involves in particular the proof that Ω is a Heyting algebra, which is non-trivial especially for the \Rightarrow operator, and then composes these properties somehow.

In the absence of simpler formulations of those steps, we were only able to investigate the behavior of the algorithm by observational means, applying it to specific derivations, as shown below.

4.2 Examples

Implication cut When applied to a simple implication cut, the algorithm does what expected.

Initial proof	Reduct
$\frac{\frac{\overline{A, A \vdash A} \text{ ax}}{A \vdash A \Rightarrow A} \Rightarrow_I \quad \frac{\overline{A \vdash A} \text{ ax}}{A \vdash A} \Rightarrow_E}{A \vdash A}$	$\frac{\overline{A \vdash A} \text{ ax}}{A \vdash A}$

Disjunction cut A disjunction cut is also properly reduced:

Initial proof	Reduct
$\frac{\frac{\overline{A \vdash A} \text{ ax}}{A \vdash A \vee A} \vee_{I_l} \quad \frac{\frac{\overline{A, A \vdash A} \text{ ax}}{A, A \vdash A \vee A} \vee_{I_r} \quad \frac{\overline{A, A \vdash A} \text{ ax}}{A, A \vdash A \vee A} \vee_{I_l}}{A \vdash A \vee A} \vee_E}{A \vdash A \vee A}$	$\frac{\overline{A \vdash A} \text{ ax}}{A \vdash A \vee A} \vee_{I_r}$

Eta expansion As the algorithm is type-directed, it is not neutral on elementary proofs, when formulas are not atomic:

Initial proof	Reduct
$\frac{\overline{A \Rightarrow B \vdash A \Rightarrow B} \text{ ax}}{A \Rightarrow B \vdash A \Rightarrow B} \text{ ax}$	$\frac{\frac{\overline{A \Rightarrow B, A \vdash A \Rightarrow B} \text{ ax} \quad \frac{\overline{A \Rightarrow B, A \vdash A} \text{ ax}}{A \Rightarrow B, A \vdash B} \Rightarrow_E}{A \Rightarrow B, A \vdash B} \Rightarrow_I}{A \Rightarrow B \vdash A \Rightarrow B}$
$\frac{\overline{A \wedge B \vdash A \wedge B} \text{ ax}}{A \wedge B \vdash A \wedge B} \text{ ax}$	$\frac{\frac{\frac{\overline{A \wedge B \vdash A \wedge B} \text{ ax}}{A \wedge B \vdash A} \wedge_{E_l} \quad \frac{\overline{A \wedge B \vdash A \wedge B} \text{ ax}}{A \wedge B \vdash B} \wedge_{E_r}}{A \wedge B \vdash A \wedge B} \wedge_I}{A \wedge B \vdash A \wedge B}$
$\frac{\overline{A \vee B \vdash A \vee B} \text{ ax}}{A \vee B \vdash A \vee B} \text{ ax}$	$\frac{\frac{\overline{A \vee B \vdash A \vee B} \text{ ax}}{A \vee B \vdash A \vee B} \text{ ax} \quad \frac{\frac{\overline{A \vee B, A \vdash A} \text{ ax}}{A \vee B, A \vdash A \vee B} \vee_{I_l} \quad \frac{\overline{A \vee B, B \vdash B} \text{ ax}}{A \vee B, B \vdash A \vee B} \vee_{I_r}}{A \vee B \vdash A \vee B} \vee_E}{A \vee B \vdash A \vee B}$

When applying iteratively the theorem once again to those proofs, one can notice that we already have reached a fixed point in the \Rightarrow and \wedge cases, while we continue reducing in the disjunction case, where we can notice that the context is abstracted via a \Rightarrow_I rule, that introduces a commutative cut:

$$\frac{\frac{\frac{\frac{A \vee B \vdash A \vee B}{ax}}{A \vee B, A, A \vee B \vdash A} \vee_{I_l}}{A \vee B, A \vdash (A \vee B) \Rightarrow (A \vee B)} \Rightarrow_I \quad \frac{\frac{\frac{\frac{A \vee B, B, A \vee B \vdash B}{ax}}{A \vee B, B, A \vee B \vdash A \vee B} \vee_{I_r}}{A \vee B, B \vdash (A \vee B) \Rightarrow (A \vee B)} \Rightarrow_I}{A \vee B \vdash (A \vee B) \Rightarrow (A \vee B)} \vee_E}{A \vee B \vdash A \vee B} \Rightarrow_E$$

5 Conclusion

Strong completeness with respect to Heyting algebras has a constructive proof, and therefore produces an algorithm for proof normalization. This argument can also be lifted to classical logic, using Boolean algebras instead, although we would have to carefully choose a classical natural deduction calculus. Obviously, this also applies to sequent calculus, in an even more straightforward way.

Our algorithm can be studied by evaluating it on specific derivations and by *Printing* the Coq function to review the generated code. However, simplifying Coq proofs, via more general inversion (Kleene) or weakening lemmas for instance, is still necessary for a more in-depth understanding.

It would also be interesting to compare the algorithm that we obtain with the ones that come from completeness with respect to Kripke structure [9,6,8], and in particular the produced proofs. One of the interest of our methodology is that we deal with disjunction (sum types) without requiring any modification of the semantics.

Semantic transformations could help in the study of the relationship between both algorithms. In particular, turning a Heyting algebra into a Kripke structure is not purely constructive [15], which may, computationally speaking, introduce control operators, as used in [8]. Applied to the particular universal Heyting algebra/Kripke structure, translations may also be more informative and constructive.

As for disjunction, we did not focus on commutative cuts, and more work is required to enhance the cut elimination algorithm, in this direction. It is theoretically possible, as we can always eliminate those cuts by translating back and forth natural deduction into sequent calculus, semantically normalizing there. But a direct study is much more preferable.

Strong completeness for higher-order logic is also within reach, which, besides giving a normalization algorithm for a powerful logic, would give another way of studying disjunction, through their higher-order encoding.

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